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Investigating the Minimal 4-Regular Matchstick Graph

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Background:

A graph is a collection of vertices and of the unordered pairs of adjacent vertices, or edges. A graph is called matchstick if there exists an embedding in the plane, such that every pair of adjacent vertices are unit distant and edges do notoverlap. If every vertex has *r* edges emanating from it, then the graph is called r-regular. The regions bounded by s-gons in a graph are called faces. The region exterior to the graph is also a face, called the outer face. Any vertex not on the outer face is called an inner vertex. An r-regular matchstick graph is called incomplete if the inner vertices all have degree r, but the vertices on the outer face have degree at most r. If every vertex on the outer face also has degree r, then the graph is called complete. To measure how incomplete a graph is, we consider:

$$\delta = r \cdot k - \sum_{v \in K} \delta(v)$$

Where K is the set of all vertices on the outer face, k is the cardinality of K, and δ(v) is the degree of the vertex, v.

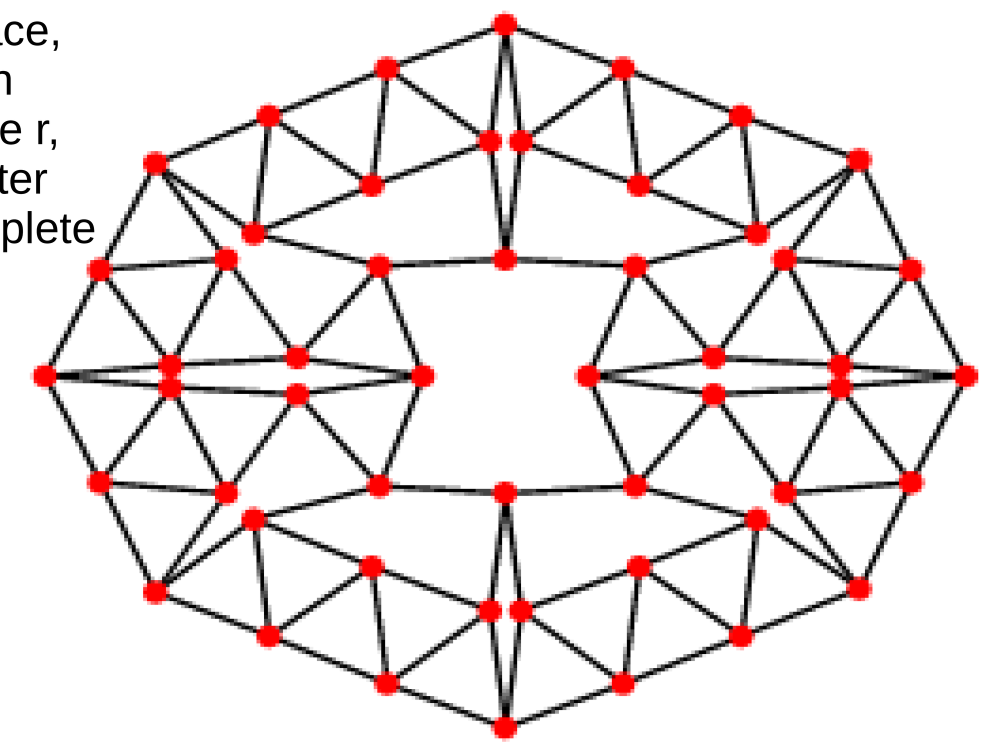

Figure 1. The smallest known 4-regular matchstick graph, first presented by H. Harborth and thus called the Harborth Graph.

The Minimal Case:

What is the minimum number of vertices, m(r), needed for an r-regular matchstick graph? It's easy to see that m(0) = 1, m(1) = 2, m(2) = 3, corresponding to a single vertex, a single edge and a triangle, respectively. Determining m(3) = 8, figure 4, is non-trivial, and m(r) is unbounded for R ≥ 5, leaving only the r = 4 case unsolved. The smallest known example of a 4-regular matchstick graph is called the Harborth Graph, Figure 1, named for Heiko Harborth who first presented it. The Harborth Graph has 52 verticies, which gives us m(4) ≤ 52. An exhaustive computer search was required to show that m(4) ≥ 34 [1]

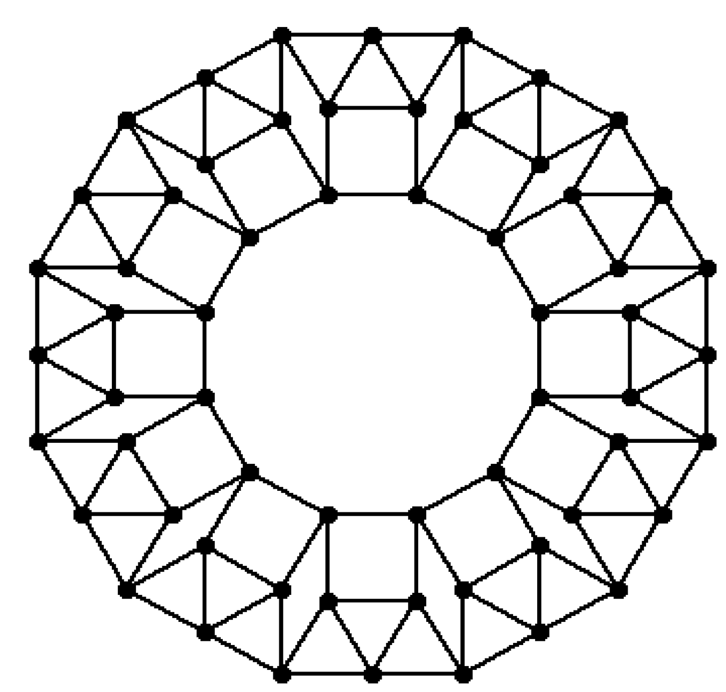

Figure 2. An non-minimal 4-regular matchstick graph

Planar Embeddings:

An embedding is one of many possible “drawings” of a graph. The two graphs in figure 3, below, are examples of different embeddings of the same graph. Neither embedding is matchstick. The left embedding fails because its edges do not maintain unit distance, while the right embedding fails to be matchstick because edges overlap.

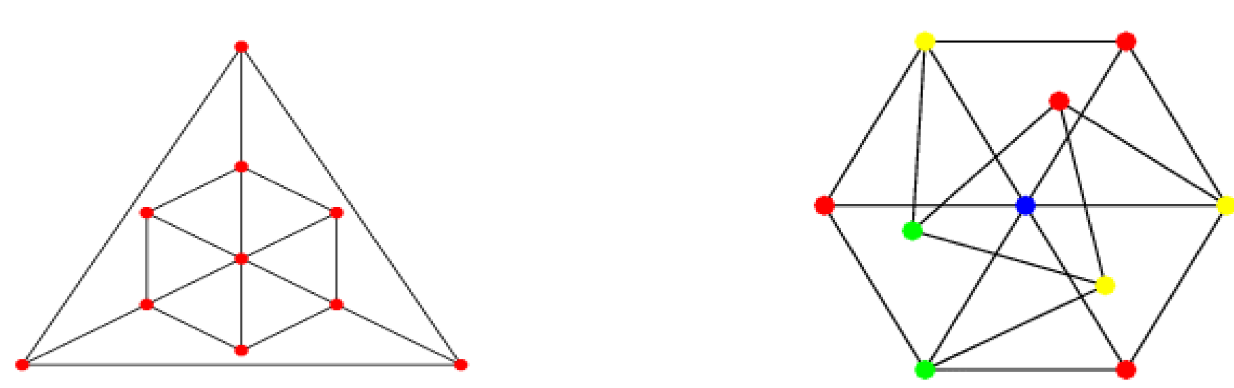

Figure 3. Two non-matchstick embeddings of the same graph. The left embedding fails to be matchstick because not all edges are unit length. The right embedding fails because edges overlap.

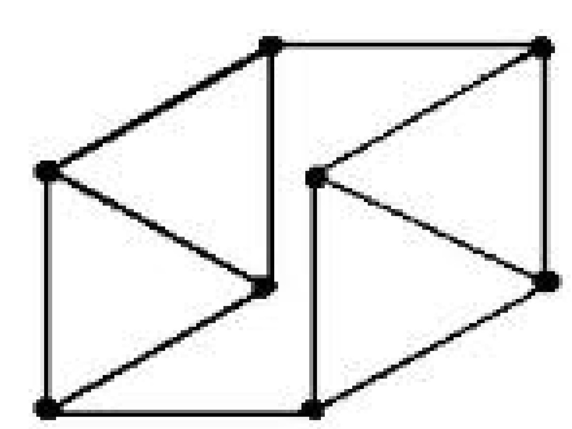

Figure 4. The minimal 3-regular matchstick graph

Planarity Restrictions:

Certain restrictions are induced due to the planar nature of matchstick graphs. Such as Euler's Polyhedron Formula: V – E + F = 2, where V is the number of vertices, E is the number of edges and F is the number of Faces. From this we can derive:

$$A_3 = 4 + k + A_5 + 2A_6 + \ldots + (k-4)A_k - 1 + \ldots$$

Where A_i is the number of i-gons bounding a face. This is a powerful formula that restricts the numbers of s-gons bounding faces.

Angle Restrictions:

- Unit length edges impies 3-gons are equilateral triangles
  - Each angle is π/3 radians
- Unit length edges also implies that any 4-gon is a rhombus
  - Neighbored angles sum to π
- 4-regularity implies at most three 3-gons are adjacent to any one vertex
  - The fourth angle at such a vertex is π radians
- Inner angles of an s-gon sum to (s-2)π, outer angles sum to (s+2)π
- Proven in [1], neighboring angles in a matchstick polygon sum to greater than π/2

Area Restrictions:

- k-gon bounding outer face implies a maximum area for inner area of graph
- Unit length edges force s-gons, where s is odd, to have a strictly positive area [1]
- If a vertex is adjacent to only 3-gons and 4-gons, then the area of the adjacent 4-gons is greater than the area of the adjacent 3-gons [1]
- If 2 of a 5-gon's vertices are adjacent to three 3-gons, then the 5-gon is an isosceles triangle
  - The area of this 5-gon is greater than two 3-gons worth of area

A useful technique for applying these restrictions is to find a chain of triangles where the majority of the edges are adjacent to the outer face. We then consider the subgraph without the triangle chain, which is bounded by a smaller k-gon. If that subgraph breaks the area argument, then neither the subgraph nor the original graph are matchstick.

Generating Graphs:

The exhaustive computer search showing m(4) ≥ 34 was performed by Dr. Sascha Kurz. A computer generated all planar graphs up to 34 vertices. Then the above restrictions were used to filter out most non-matchstick graphs. However a few graphs slipped through. To show these graphs weren't matchstick graphs, Dr. Kurz used brute force methods, such as calculating the exact coordinates and showing the distance between two adjacent vertices could not be one unit. I focused my efforts this summer on finding easier ways to show these graphs were not matchstick.

References:

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