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Symmetry Methods and Self-Similar Solutions to Curve Shortening

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Abstract

Curve shortening is a geometric process that continually evolves a curve based on its curvature. Self-similar solutions to the curve shortening equation maintain their form throughout the process, though they can be scaled, translated, or rotated. These self-similar solutions correspond to the invariant solutions of the symmetry method for solving differential equations.

1. Symmetry Methods

Symmetry methods are a technique for solving differential equations.

- A symmetry for a differential equation maps solutions to solutions, for example by scaling or translating.
- The goal is to use a symmetry to turn the differential equation into a form that is easier to solve by normal methods (e.g., separation of variables).
- Symmetries exist in one-parameter families that produce flows where solutions are continuously mapped to solutions (as the value of the parameter changes).
- Example [3]: The scaling transformation $(x, y) \rightarrow \left(\alpha^2 x, \alpha^2 y\right)$ is a symmetry flow for the differential equation $\frac{dy}{dx} = y^2 + 2y + 1$.

The green flow lines show the change in the blue solutions as $\alpha$ changes. Two invariant solutions are shown in red.

- An invariant solution to a differential equation is one that is mapped to itself in the symmetry, i.e., it is invariant in the symmetry.
- In order to find invariant solutions to a symmetry, we use what are called canonical coordinates. Converting to canonical coordinates results in an equation that is much easier analyze and if we’re lucky, solve.
- Once a solution is found for the transformed equation, we can easily transform back to the original coordinates using the definitions for our canonical coordinates.

2. Symmetry Generators

- Symmetries can be expressed in one of two ways:
  - as $(\xi, \eta)$ given as functions of the old coordinates $(x, y)$ and a parameter $\epsilon$.
  - as a symmetry generator $X = \partial_x + \eta(x, y)\partial_y$ where $\xi$ and $\eta$ are functions of $x$ and $y$ defined by
    \[ \frac{\partial}{\partial x} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \]
- All symmetries for a differential equation, $\frac{dy}{dx} = \omega(x, y)$ must satisfy what is known as the symmetry condition
  - The full symmetry condition is used with functions $\xi$ and $\eta$ defined by
    \[ \frac{\partial}{\partial x} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \]
- For symmetry generators, we linearize this condition around $x = 0$.
- In order to find symmetries, we use the linearized condition because the linear equations that result are typically easier to solve.

3. Curve Shortening

- Curve shortening is a geometric evolution that when given a curve, the curve continually evolves based on the curvature [2].

4. Curve Shortening for the Graph of a Function

- As shown in [1], the first option of looking directly at the curve as the graph of a function $u(x)$ results in the differential equation $\frac{d\bar{u}}{d\bar{x}} = \frac{1}{1 + \bar{u}^2}$.

5. Curve Shortening Applied to the Curve

- For the curve shortening system, our independent variables are time $t$ and the arbitrary parameter $\mu$. The dependent variables are the curvature $k$ and $x = \frac{1}{\sqrt{1 + k^2}}$.
- We are able to reduce the following system of differential equations from the original curve shortening equation:
  \[ \frac{dk}{dt} = \frac{2k}{\sqrt{1 + k^2}} \frac{1}{\sqrt{1 + k^2}} \frac{dy}{dx} + k^2 \]
- From the linearized symmetry condition, we get a system of 31 determining equations.
- From this system, we are able to deduce:
  \[ \xi = C(p), \quad \tau = -2\sqrt{p} \chi, \quad \chi = c_1 \chi, \quad \text{and} \quad \eta = c_2 C(p) + c_1 \]
  where $c_1$ and $c_2$ are constants and $C$ is any differentiable function.
- The above generator describes all possible symmetries for our system, so the next step was to find invariant solutions for particular generators. The generator that we analyzed was $X = p \partial_p + 2h \partial_h$, where $h$ is the vector valued function for the curve.
- This generator results in the canonical coordinates $\tau = \frac{1}{\sqrt{1 + h^2}} \left(\frac{2\sqrt{p}}{\sqrt{1 + h^2}} + \frac{1}{2} h^2\right)$.
- Once completely converted to canonical coordinates, the system turns into the following:
  \[ G' = \frac{1 + h^2}{1 + h^2} \]
  \[ H' = \frac{1}{1 + h^2} \frac{1}{1 + h^2} \left(2h \partial_h + 4h \partial_h + 4h \partial_h \right) \]
- Though made difficult with the factor of $\tau^{-1}$, the next step would be to analyze these equations. However, this was beyond the scope of this project for the summer.

References