Gravitational wave behavior at a vacuum-matter interface

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An Honors thesis by

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Abstract

In classical electrodynamics, boundary conditions of the E and B fields are derived from Maxwell's equations, which are used to derive the Fresnel equations describing the behavior of a wave at an interface between media with given indices of refraction. Though electrodynamics and gravity are in some instances strikingly analogous, boundary conditions in general relativity are somewhat more opaque. We will see that while while continuity of the metric must be true in general, discontinuity of the extrinsic curvature of spacetime, while allowed by the Einstein field equations, results in a singularity in the energy-momentum tensor. This singularity is interpreted as a surface mass density. Unlike in electrodynamics, there is an additional refractive effect of the spacetime metric. Its origin considered, a gravitational refractive index will be treated similarly to the electromagnetic refractive index. Attempts to derive gravitational "Fresnel equations" follow.

1 Introduction

Isaac Newton's law of universal gravitational attraction, the inverse square law of gravitational force between point masses, is a useful tool for making predictions in weak gravitational fields. However, it fails to describe certain astronomical observations, predicts the worrisome instantaneous action at infinite distance, and is incompatible with special relativity. In 1915, Albert Einstein proposed his general theory of relativity, the fully relativistic theory of gravity that reconceptualizes gravity as the result of a curved spacetime, and predicts the existence of gravitational waves. After the recent detection of gravitational waves by the Laser Interferometer Gravitational-Wave Observatory (LIGO) detectors in September 2015 [1], almost exactly a hundred years after Einstein first predicted their existence, gravitational waves, general relativity, and Einstein were the subject of many popular science magazine articles – for a couple months, anyway. While the general public may not retain much interest, the physics community hopes that gravitational wave astronomy will prove to be a useful astronomical tool, beginning a new epoch in astronomy and making groundbreaking discoveries possible.

The existence and nature of gravitational waves (GWs) was for many years the subject of contentious debate. Einstein himself changed his mind several times, believing at first that GWs exist in analogy to light waves in electromagnetic theory (EM). After working on this problem with his assistant Nathan Rosen, Einstein wrote in a letter to physicist Karl Schwarschild that he believed GWs not to exist, probably in relation to the lack of negative matter [2]. Einstein and Rosen tried to publish these findings in *The Physical Review*, but Einstein was angered that his work should be subjected to peer review before publication and decided to publish the paper elsewhere. Had he seriously considered this initial criticism, the errors in the calculations might have been corrected before being published, and Einstein would have sooner realized that GWs must in fact exist. To his credit, after eventually coming to this conclusion Einstein thanked the original reviewer [2]. The existence of gravitational waves assured, physicists began to wonder how they might be detected.

Joseph Weber likely became interested in trying to detect GWs after attending the first GR1 conference in 1957 at Chapel Hill (the first major general relativity

conference that continues to this day in high prestige). In 1969, he claimed to have detected GWs using a large aluminum cylinder. He measured the mechanical vibration of the cylinder using an array of piezoelectric crystals, and claimed that these vibrations were caused by gravitational waves passing through the cylinders [2]. However, multiple people independently concluded Weber's results imply that in the Milky Way, so much mass is being converted into energy in the form of gravitational radiation that the galaxy must not even exist [2]. Weber's claimed results were clearly not credible. Some, however, took after Weber in the attempt to detect GWs. The use of Weber-type detectors persisted for a while, until the idea of an interferometer detector gained popularity. While it's unknown for sure who first thought of using a laser interferometer to detect GWs, the idea stuck. It took about fifty more years after Weber for the first true detection by LIGO to be made.

Along with much of the general public, my interest in GWs was piqued by their detection. My motivation for studying GWs originated earlier, though, during an exam in my junior-year EM class. In analogy to the electric field, we were asked to derive a boundary condition on the Newtonian gravitational field. I concluded in perfect analogy to the boundary condition on the electric field that there is a discontinuity given by the surface mass density on the boundary. I was told after the exam that this was wrong because surface mass densities don't exist; the gravitational field must be continuous everywhere. I couldn't see why surface mass densities shouldn't exist. If we posit the existence of surface charge densities, discounting the existence of surface mass densities seems inconsistent. Nothing in the derivation of the boundary conditions on the Newtonian gravitational field implies that there are no surface mass densities. Perhaps there is such an implication in GR. When considering investigating this question myself, I thought that only investigating boundary conditions for the gravitational field wouldn't be very difficult, and after acquiring the boundary conditions, it should be easy to see how a GW refracts and reflects at an interface. This became the topic of my thesis.

The problem is hardly as simple as in junior-level EM (if that can be taken as a baseline of simplicity), nor as simply as I imagined. First of all, a true answer will only be found in general relativity (GR), not Newtonian gravity. Newtonian gravity is a vector field theory with one governing field equation, while gravity in GR is

a second-rank tensor field theory and the field equations are ten coupled nonlinear partial differential equations. The methods used in EM cannot be directly applied to GR as to Newtonian gravity. Much of my thesis work therefore involved learning whatever GR necessary to answer the problem at hand, and my knowledge of GR is ad hoc and incomplete. Throughout this paper, a college physics student's knowledge of math, classical mechanics, special relativity and basic EM are assumed. Some differential geometry will be developed when necessary for GR. To aid the reader, EM will be used in analogy to GR whenever applicable. This development should give the reader an idea of my own learning process, without having to do the actual research.

Because of the similarities between EM and gravity, both Newtonian and (especially linearized) GR, we first consider the problem in EM and find the behavior of an electromagnetic wave (EM wave) incident on a planar medium. The origin of the electromagnetic index of refraction is also discussed. After a brief introduction to GR, boundary conditions in GR are discussed (significantly, surface mass densities may indeed be admitted and give valid solutions to the field equations). A gravitational index of refraction is also introduced. The goal is to use these boundary conditions to derive the behavior of a gravitational wave at an interface. Some naive attempts are made, some blind alleys followed. Pressing on, we may find the answer we sought.

Or maybe not.

2 Do it in EM

Let's begin by first tackling the problem with EM waves in classical electrodynamics, where the solution is well known and fairly straightforward. We will first show that the field equations allow EM waves. Using the fields equations, we derive boundary conditions on the fields, and then use these to determine the behavior of an EM wave at an interface between two media with given indices of refraction. Finally, we calculate the index of refraction of a thin sheet of charges due to induced dipole radiation in the interface. This knowledge will be instructive when trying to repeat this procedure with gravitational waves in GR.

2.1 The wave equation

The field equations in EM are the four Maxwell's equations,

$$
\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad\qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
$$
$$
\nabla \cdot \vec{B} = 0 \qquad\qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{2.1}
$$

(taken from Jackson's *Classical Electrodynamics* [3]) for the divergence and curl of the electric and magnetic fields $\vec{E}$ and $\vec{B}$ (the Lorentz force law $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ is often included as a fundamental equation in EM, which is unnecessary in this context as Maxwell's equations uniquely determine $\vec{E}$ and $\vec{B}$). From Maxwell's equations, we can show that the $\vec{E}$ and $\vec{B}$ fields can support waves. Taking the simplest case, write Maxwell's equations in vacuo by letting $\rho = 0$ and $\vec{J} = \vec{0}$ in Eqs. (2.1). Then taking the curl of the curl of $\vec{E}$ and using the curl of $\vec{B}$ equation we get

$$
\begin{aligned}
\nabla \times (\nabla \times \vec{E}) &= \nabla \times -\frac{\partial \vec{B}}{\partial t} \\
&= -\frac{\partial}{\partial t}(\nabla \times \vec{B}) \\
&= -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\vec{E}. \tag{2.2}
\end{aligned}
$$

We can use a vector identity $(\nabla \times \{\nabla \times \vec{u}\} = \nabla\{\nabla \cdot \vec{u}\} - \nabla^2\vec{u})$ and the divergence of $\vec{E}$ equation to simplify the left hand side to

$$\begin{aligned}
\nabla \times (\nabla \times \vec{E}) &= \nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E} \\
&= \nabla(\rho/\epsilon_0) - \nabla^2\vec{E} \\
&= -\nabla^2\vec{E}.
\end{aligned} \tag{2.3}$$

Equating Eq. (2.2) to Eq. (2.3), we can write

$$\nabla^2\vec{E} = \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\vec{E}. \tag{2.4}$$

Comparing this to the classical d'Almbertian wave equation in some field A,

$$\Box A \equiv \frac{1}{v^2}\frac{\partial^2}{\partial t^2}A - \nabla^2 A = 0,$$

we can see that, at least in vacuum, the electric field can support waves, and they propagate at speed $v = 1/\sqrt{\mu_0\epsilon_0} \equiv c$.

A similar procedure, taking the curl of the curl of $\vec{B}$, reveals the same wave equation in the magnetic field $\vec{B}$. One can derive several other results about the $\vec{E}$ and $\vec{B}$ waves from Maxwell's equations; most importantly for our purposes is that they're both transverse. They also happen to be perpendicular to each other and in phase.

2.2 Boundary conditions

Now that we know light exists (*phew*), how does it behave when crossing an interface? What conditions, if any, do Maxwell's equations impose on $\vec{E}$ and $\vec{B}$ at an interface? To derive these boundary conditions, we would need to rewrite Eqs. (2.1) in their integral forms using Gauss' theorem and Stokes' theorem. To get the boundary conditions necessary later, we will only need the curl of $\vec{E}$ and the curl of $\vec{B}$ in their integral forms (in vacuo):

$$\oint_P \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a}$$

$$\frac{1}{\mu\epsilon}\oint_P \vec{B} \cdot d\vec{l} = \frac{d}{dt}\int_S \vec{E} \cdot d\vec{a} \tag{2.5}$$

The left side of these equations is a path integral around a closed path, and the right side is the surface integral of the area enclosed by that closed path. The utility we will exploit is that the path can be *any* closed loop, anywhere we like. We therefore choose to make it a rectangular loop perpendicular to a boundary between two regions, with the same height h above the boundary on either side.
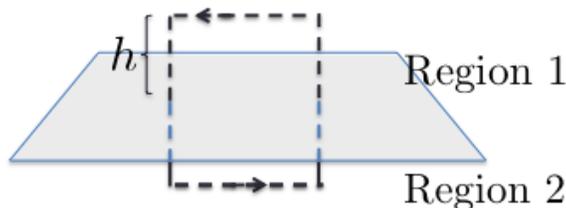

Figure 1: Stokes loop symmetric across a boundary between two regions.

Taking the path integral of $\vec{E}$ around this path, the ends of the rectangle contribute nothing to the path integral, so

$$\oint_P \vec{E} \cdot d\vec{l} = h \; \vec{E}_{\text{Region 1}}^{\parallel} - h \; \vec{E}_{\text{Region 2}}^{\parallel} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{a}. \tag{2.6}$$

We take the limit $h \to 0$, shrinking the loop down to the boundary, so the area integral vanishes. This gives

$$\vec{E}_{\text{Region 1}}^{\parallel} = \vec{E}_{\text{Region 2}}^{\parallel}. \tag{2.7}$$

Thus the component of $\vec{E}$ parallel to any boundary must be *continuous* across that boundary. We consider the $\vec{B}$ integral in Eq. (2.5) similarly, where we shrink the loop to make the area integral vanish. But we must be careful about the constants in front of the path integral. If the magnetic permeability is the same on both sides of the boundary, we could divide it through into the zero, but this is not true in general. Accounting for a different permeability on either side of the boundary, we then get

$$\frac{1}{\mu_1} \vec{B}_{\text{Region 1}}^{\parallel} = \frac{1}{\mu_2} \vec{B}_{\text{Region 2}}^{\parallel}. \tag{2.8}$$

If the permeabilities are the same ($\mu_1 = \mu_2$), they simply cancel. If not, as is generally true, we have found that the component of $\vec{B}$ parallel to the boundary is *discontinuous*, with the discontinuity given by the permeabilities on either side of the boundary [3].

The equations for the divergence of $\vec{E}$ and $\vec{B}$ can be used to find the boundary conditions on $E^\perp$ and $B^\perp$, the components of $\vec{E}$ and $\vec{B}$ perpendicular to the boundary, but we won't need these in the derivation that follows.

2.3   Fresnel equations

The goal of this section is to use the boundary conditions Eqs. (2.7) and (2.8) to determine the behavior of an EM wave incident upon a boundary between two media with indices of refraction n_1 and n_2. For $n_1 \neq n_2$, we expect some reflection and refracted transmission. We begin knowing the kinematic properties: the "law of reflection," that the angle of incidence equals the angle of reflection; and Snell's law, $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$. It's also intuitive that the incident, reflected, and transmitted waves lie in a plane [3]. These three laws are generally true, for EM waves and, we expect, for gravitational waves.

A note on the index of refraction: We define it as $n \equiv c/v = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$, and it can be determined experimentally without much difficulty. For instance, in air $n \approx 1$, so we can easily calculate an index of refraction from Snell's law by shining a laser into a material and measuring the angles of incidence and refraction. We will therefore treat indices of refraction as known material constants.

After deriving Eq. (2.4), $\vec{E}$ was left unspecified. It now becomes necessary to give a generic solution to this wave equation. For a monochromatic plane electric wave,

$$\vec{E} = \vec{E_0}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \tag{2.9}$$

where $\vec{E_0}$ points in the polarization direction and its length is the amplitude of the wave; $\vec{k}$, called the wave vector, points in the direction of propagation and its length k is the wave number (i.e. $k\,v = \omega$). We will just work with the electric wave because

the $\vec{B}$ wave anywhere

$$\vec{B} = \sqrt{\mu\epsilon}\,\frac{\vec{k} \times \vec{E}}{k}$$

is easily written in terms of $\vec{E}$ [3]. So if we know the $\vec{E}$ wave everywhere, we also know the $\vec{B}$ wave everywhere. Taking Eq. (2.9) as the incident wave, we posit that the reflected and transmitted waves have the same form, but perhaps with different polarization and wave vectors:

$$\text{reflected} \qquad \vec{E'} = \vec{E'_0}e^{i(\vec{k'}\cdot\vec{r}-\omega t)} \qquad\qquad (2.10)$$

$$\text{transmitted} \qquad \vec{E''} = \vec{E''_0}e^{i(\vec{k''}\cdot\vec{r}-\omega t)} \qquad\qquad (2.11)$$

Since the incident and reflected waves are in the same medium, $n = n'$. The boundary conditions must hold everywhere on the interface for all time. This requires that at the interface

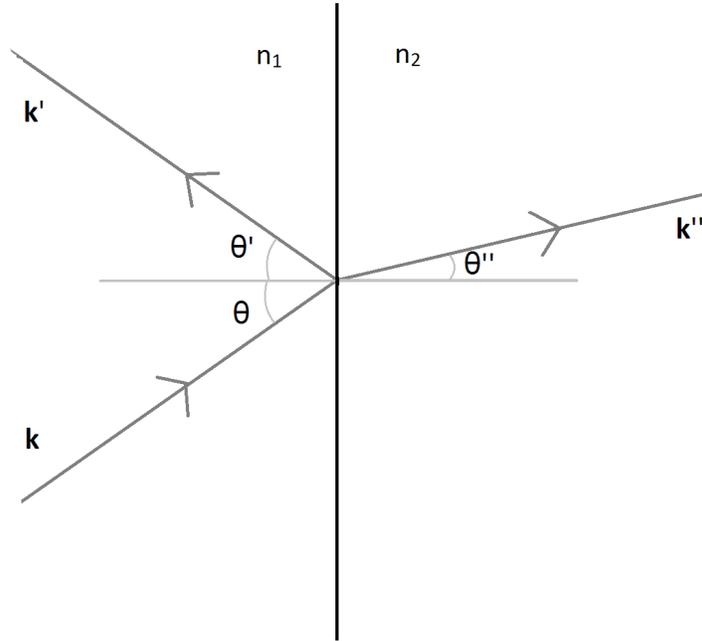$$\vec{k}\cdot\vec{r} = \vec{k'}\cdot\vec{r} = \vec{k''}\cdot\vec{r}. \qquad\qquad (2.12)$$

Figure 2: Incident, reflected, and transmitted EM waves at an interface.

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Then we can plug Eqs. (2.9)–(2.11) into the boundary conditions and cancel the exponential factors to get

$$\vec{E_0}^{\parallel} + (\vec{E_0}^{\parallel})' = (\vec{E_0}^{\parallel})'', \tag{2.13}$$

$$\frac{1}{\mu_1}(\vec{B_0}^{\parallel} + (\vec{B_0}^{\parallel})') = \frac{1}{\mu_2}(\vec{B_0}^{\parallel})''. \tag{2.14}$$

From here we can't continue with full generality; we have to make a choice of polarization. Choose (for no particular reason) $\vec{E_0}$ parallel to the interface. Then the boundary conditions become

$$E_0 \, cos(\theta_I) + E_0' \, cos(\theta_R) = E_0'' \, cos(\theta_T), \tag{2.15}$$

$$\frac{1}{\mu_1 \epsilon_1}(E_0 - E_0') = \frac{1}{\mu_2 \epsilon_2}E_0''. \tag{2.16}$$

Remembering the law of reflection, Eq. (2.15) becomes

$$E_0 + E_0' = \frac{cos(\theta_T)}{cos(\theta_R)}E_0''. \tag{2.17}$$

Solving Eqs. (2.16) and (2.17) for E_0' and E_0'', we get

$$E_0' = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)E_0 \qquad E_0'' = \left(\frac{2}{\alpha + \beta}\right)E_0, \tag{2.18}$$

where

$$\alpha \equiv \frac{cos(\theta_T)}{cos(\theta_R)} \qquad \beta \equiv \frac{\mu_1 n_2}{\mu_2 n_1}$$

[3]. These are called Fresnel's equations, and they together with Snell's law and the law of reflection tell us everything about the EM wave as it passes through an interface. As stated above, these are the Fresnel equations for polarization parallel to the interface. The other basis polarization state would be polarization perpendicular to the interface, and every possible polarization can be described as a superposition of these two states. To find the Fresnel equations for the perpendicular polarization state, we would need the other two boundary conditions from Sect. (2.2) that I did not derive, using the other two Maxwell's equations. What's important for moving

on to GWs is the procedure of using the field equations to derive boundary conditions that the wave must satisfy at the interface, choosing a polarization, and then using these boundary conditions to determine the scattering behavior of the wave at such an interface.

2.4 EM refractive index

In the previous discussion of the Fresnel equations, the index of refraction was introduced as an inherent material property which we could determine experimentally. In practical optics, this is generally true. It makes more sense to experimentally determine the index of refraction of a given material than to measure all the material and atomic constants and try to calculate the index of refraction from these. However, we are of late unable to perform such experiments with gravitational waves, so we will need to know how to derive the gravitational index of refraction mathematically. As per the mode of operation in this chapter, we will see how to do this calculation for familiar EM waves. We will find the contribution to radiation from a single electric dipole in the interface, then use the principle of superposition to add up the total radiation from this interface.

Consider a planar electromagnetic wave that passes through a thin plate of matter at $z = 0$ in the x-y plane. Following Feynman's *Lectures* [4], the wave after the plate must be the superposition of the source wave and waves produced by the interaction between the source wave and the plate. The electric wave from a distant source

$$\vec{E}_s = \vec{E}_0 \ e^{i\omega(t-z/c)} \tag{2.19}$$

passes through the plate of thickness Δz with index of refraction n, taking the additional time $\Delta t = (n\Delta z/c \ - \ \Delta z/c)$ to pass through the space occupied by the plate. Replacing t in Eq. (2.19) by $t - \Delta t$, $\vec{E}$ after the plate is

$$\vec{E}_{after \ plate} = \vec{E}_0 \ e^{i\omega[t-(n-1)\Delta z/c \ - \ z/c]}$$
$$= e^{-i\omega(n-1)\Delta z/c}\vec{E}_0 \ e^{i\omega(t-z/c)} \tag{2.20}$$

which means that the wave after the plate is given by just the wave from the source multiplied by a new exponential term. Because the plate is thin (Δz is small), we

can use the Taylor expansion of the new exponential and take the first two terms to get

$$e^{-i\omega(n-1)\Delta z/c} = 1 - i\omega(n-1)\Delta z/c. \tag{2.21}$$

Then

$$\vec{E}_{after\ plate} = \vec{E}_0\ e^{i\omega(t-z/c)} - \frac{i\omega(n-1)\Delta z}{c}\vec{E}_0\ e^{i\omega(t-z/c)} \tag{2.22}$$

$$= \vec{E}_s + \vec{E}_a \tag{2.23}$$

where $\vec{E}_a$ must be the electric wave produced by the plate. Note that the new wave is proportional to the magnitude and frequency of the source wave, and the index of refraction and thickness of the plate, as we might expect.

So what is n? Because the source is very far away, $\vec{E}_s$ has the same value everywhere on the plate [4]. At the plate, where $z = 0$,

$$\vec{E}_s = \vec{E}_0\ e^{i\omega t}.$$

This field acts as a driving force ($\vec{F} = q\vec{E}$) on the electrons in the plate, making them oscillate with resonant frequency ω_0. Treating each electron as a classical simple harmonic oscillator, the equation of motion for one at the origin is

$$m_e(\frac{d^2 x}{dt^2} + \omega_0^2 x) = q_e E_0 e^{i\omega t}.$$

Solving this differential equation, we get the familiar solution

$$x(t) = x_0 e^{i\omega t} \tag{2.24}$$

where

$$x_0 = \frac{q_e E_0}{m_e(\omega_0^2 - \omega^2)}. \tag{2.25}$$

Since all the electrons in the plate move the same way, we now know how all the electrons in the plate are moving. To calculate the radiation produced by a sheet

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of oscillating charges, we can calculate the radiation from one oscillating charge and integrate over the surface of the plate to find the total contribution from every oscillating electron.

The acceleration of each charge is then

$$\frac{d^2[x(t)]}{dt^2} = -\omega^2 x_0 e^{i\omega t}. \tag{2.26}$$

Replacing t with the retarded time $(t-r/c)$ where r is the distance from the oscillating charge to the point, this gives $-\omega^2 x_0 e^{i\omega(t-r/c)}$. Then the electric field from one electron at a point P very far away from the plane is

$$E_{one\ charge}(P) = \frac{q_e}{4\pi\epsilon_0 c^2} \frac{\omega^2 x_0 e^{i\omega(t-r/c)}}{r}. \tag{2.27}$$

Considering all of the electrons oscillating in the plate, they will each radiate spherically as described by Eq. (2.27). These waves will interfere destructively, canceling each other out everywhere except at the wavefront [3, 4]. Integrating Eq. (2.27) over the entire plate in polar coordinates, where $s^2 = r^2 - z^2$ and σ is the surface charge density,

$$E_{total}(P) = \int_0^{2\pi} \int_0^\infty \frac{\sigma}{4\pi\epsilon_0 c^2} \frac{\omega^2 x_0 e^{i\omega(t-r/c)}}{r}\ s\ ds\ d\phi \tag{2.28}$$

$$= -\frac{\sigma}{2\epsilon_0 c} i\omega x_0 e^{i\omega(t-z/c)}. \tag{2.29}$$

Substituting x_0 from Eq. (2.25), the electric field from the plate of oscillating electrons is

$$E_a = -\frac{\sigma}{2\epsilon_0 c} i\omega \left(\frac{q_e E_0}{m_e(\omega_0^2 - \omega^2)}\right) e^{i\omega(t-z/v)} \tag{2.30}$$

which depends on the expected source wave magnitude as well as charge density and atomic properties. Comparing this to the expression of E_a in Eqs. (2.22) and (2.23), these will only be identical if

$$(n-1)\Delta z = \frac{\sigma q_e}{2\epsilon_0 m_e(\omega_0^2 - \omega^2)}.$$

Surface charge density is equal to $\rho \Delta z$, where ρ is the volume charge density, so we cancel the Δz's to get

$$n = 1 + \frac{\rho q_e}{2\epsilon_0 m_e (\omega_0^2 - \omega^2)}. \tag{2.31}$$

As expected, the index of refraction depends on material and atomic properties. We can also see that if there are no charges present, as in a vacuum, the index of refraction will just be identical to one. Note that this analysis is only valid for n close to 1, as in a very low density cloud of gas, so that the oscillating electrons have approximately no effect on each other. Very small plate thickness is also required to use the Taylor expansion of the exponential in Eq. (2.21) [3, 4]. That may seem very restrictive, but this will actually make a useful comparison for the GW case in GR.

3 Features of GR

Now that we thoroughly understand how to determine EM wave behavior at an interface, how can we apply this to gravitational waves? Gravity is analogous to EM in some ways, and we will try to exploit these similarities. However, on the whole GR is very different from EM both mathematically and physically. We will need some new tools to tackle this. In this section, we will dive *slowly* into GR.

3.1 Similarities with EM

As stated in Sect. (1), EM is a vector field theory – $\vec{E}$ and $\vec{B}$ are vectors. In Newtonian gravity, the gravitational field is also a vector. Is this significant? Compare the Newtonian gravitational field from a point mass to the electric field from a point charge,

$$\vec{g} = G\frac{m}{r^2}\hat{r}, \qquad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}. \tag{3.1}$$

These are quite apparently similar. One depends on the mass, which can only be positive, the other the charge, which may be positive or negative – and they have different constants of proportionality (convinced yet?) – but in form they are identical! The Newtonian gravitational force looks completely analogous to the Coulomb force, except that the Coulomb force may be attractive or repulsive, while the Newtonian gravitational force may only be attractive. From Eq. (2.1), $\nabla \cdot \vec{E} = \rho/\epsilon_0$. Just by looking at Eq. (3.1), we can immediately write down a similar equation for the divergence of $\vec{g}$:

$$\nabla \cdot \vec{g} = -4\pi G\rho, \tag{3.2}$$

where ρ here is mass density. If there's one gravitational "Maxwell's equation," one might imagine there are gravitational formulations for all four. But if we take the curl of the Newtonian $\vec{g}$ field,

$$\nabla \times \vec{g} = Gm(\nabla \times \frac{1}{r^2}\hat{r}) = \vec{0}. \tag{3.3}$$

Comparing to the curl of $\vec{E}$ from Eq. (2.1), this should be a (nonzero) time derivative of a gravitational field analogous to the magnetic field. But Newtonian gravity makes no mention of a gravitomagnetic field. Oliver Heaviside hypothesized the existence of a gravitomagnetic field such that all four Maxwell's equations could be formulated for the Newtonian gravitational field and this gravitomagnetic field (before special relativity)[5], which GR also predicts, and which has been recently observed. This additional field does not solve the problems of Newtonian gravity, though. It would still be inconsistent with special relativity, still contain the worrisome instantaneous action at infinite distance, and still not be able to predict the phenomena that GR will prove able to.

Before fully introducing GR, we can qualitatively discuss some other similarities regarding waves. As mentioned above, EM waves are transverse and propagate at speed c in vacuum, and these are both true of GWs as well (we don't think of c as *the speed of light* in vacuum so much as the speed of *any* massless propagation in vacuum). Both have two polarization states but, using the basis states for EM waves described in Sect. (2.3), these polarizations are qualitatively dissimilar. This is most easily understood seen by picturing the effect of a passing wave on a collection of masses or charges, respectively.
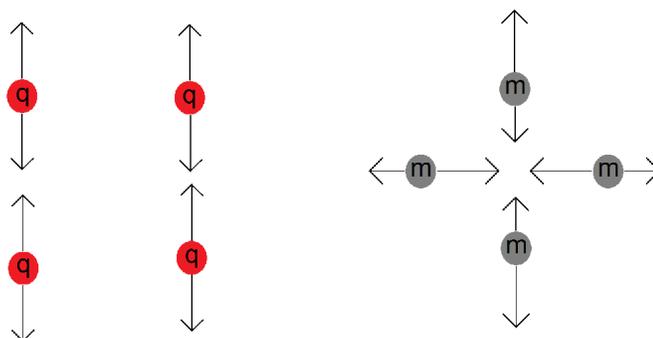

Figure 3: The effect of an incident EM wave perpendicular to a sheet of point charges, and of an incident GW perpendicular to a sheet of point masses.

While charges in the path of an EM wave all oscillate together, masses in the path of a GW move in and out opposite each other (as one can find in any GR textbook such

as Franklin's [6] or d'Inverno's [7]). This would not be possible if the gravitational field were a vector field.

Indeed, in GR the Newtonian gravitational field is replaced by the metric tensor, written as a matrix. This metric does not describe forces in the way we are used to in a vector field theory. We recover the electric force acting on a charge by multiplying by the electric field it's in, or the Newtonian gravitational force on a mass by multiplying by the gravitational field it's in. However, just multiplying the metric tensor by a mass is not particularly meaningful. The metric encodes all the information about the curvature of spacetime due to some energy distribution, and it's an object's motion along this curved background which an observer might interpret as a force in the Newtonian picture. However, in the relativistic picture, the gravitational "force" is simply motion along straight lines (geodesics) which, on a curved spacetime background, results in apparently curved motion. We use the metric to calculate these geodesics, and anything else we might wish to know about a particular system.

3.2   Geometry in four-dimensional spacetime

Now for some math. I've stated the metric is a tensor, and many other objects in GR are also tensors. We will introduce some utilitarian tensor calculus and differential geometry, and quickly apply this to physics lest we fall into the rabbit hole of mathematical formalism.

All tensors have a "rank," which is indicated by its indices. Rather trivially, rank 0 tensors are simply scalars. Rank 1 tensors have one index and are more commonly called vectors. Tensors are a generalization of scalars, vectors, and matrices to objects of arbitrary rank. An example of a rank 1 tensor is this vector in three dimensions,

$$\vec{x} = x^a = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

where f, g, h are functions of the coordinates $y^b = (x, y, z)$. A rank 2 tensor has another index, and can be written as a matrix. Tensors above rank 2 are cumbersome

to write out, as a rank 3 tensor would be a row of matrices, and a rank 4 tensor would be a matrix of matrices. When we have to write out the terms of such tensors, we write out the matrices individually, or just write out the nonzero terms, or just write out the terms of interest. Any tensor with four indices $A^{\alpha\beta}{}_{\gamma\delta}$ is called a rank 4 tensor. The "type" of this tensor is (2,2), because two of its indices are up (contravariant) and two are down (covariant). The particular meaning of a tensor's type isn't too important for us, as long as we remember that it does matter whether the indices are up or down.

Consider now a four-dimensional surface given by $z = f(t, x, y)$. By the chain rule, the total differential of this surface is

$$dz = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Similarly, if we take the differential of our vector $x^a = x^a(y^b)$, this is

$$dx^a = \frac{\partial}{\partial x}\begin{pmatrix} f \\ g \\ h \end{pmatrix} dx + \frac{\partial}{\partial y}\begin{pmatrix} f \\ g \\ h \end{pmatrix} dy + \frac{\partial}{\partial z}\begin{pmatrix} f \\ g \\ h \end{pmatrix} dz$$

$$= \sum_{b=1}^{3} \frac{\partial}{\partial y^b}\begin{pmatrix} f \\ g \\ h \end{pmatrix} dy^b$$

$$= \sum_{b=1}^{3} \frac{\partial x^a}{\partial y^b} dy^b.$$

We can clean this up a bit by using the Einstein summation notation, in which we simply drop the summation sign. Then

$$dx^a = \frac{\partial x^a}{\partial y^b} dy^b \tag{3.4}$$

where it is implied that we are to sum over the repeated index. As a rule, Greek letter indices will run from 0 to 3, while Latin letter indices will run from 1 to 3. If our 3 vector were covariant (i.e. the index in the subscript), Eq. (3.4) would be

$$dx_a = \frac{\partial y^b}{\partial x^a} dy_b. \tag{3.5}$$

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Contravariant tensors are defined as tensors which transform under a change of coordinates according to Eq. (3.4), and covariant tensors are defined as tensors which transform under a change of coordinates according to Eq. (3.5), where a and b could really be any number of dimensions, not restricted to 3. We will further economize by using either of the equivalent notations,

$$\frac{\partial x^a}{\partial y^b} \equiv x^a_{,b} \equiv \partial_b x^a$$

which we call the "ordinary derivative." The comma notation will be used when an ordinary derivative appears in an equation, and I may use the partial notation when explicitly computing an ordinary derivative. If we were to plug $\partial_b x^a$ into Eq. (3.4), we would see that the ordinary derivative of a tensor is not itself a tensor, because it doesn't transform according to Eq. (3.4). This can be explained geometrically by the fact that the basis vectors of some set of coordinates x^a are not necessarily constant in space. Take the spherical coordinate basis vectors $(\hat{r}, \hat{\phi}, \hat{\theta})$ for example. We can, however, define several different derivatives which do transform like a tensor. The only one we need concern ourselves with now is called the "covariant derivative." For a tensor x^a, the covariant derivative is

$$x^a_{;b} = x^a_{,b} + \Gamma^a_{cb} x^c \tag{3.6}$$

which appears deceptively simple. (Again, the repeated index c implies that c is being summed over in the second term on the right side. Also, since c is a Latin letter, the implied summation in this expression is from 1 to 3.) This covariant derivative is a tensor, and it's just the ordinary derivative plus some rank 3 gamma term times the tensor we're differentiating. This gamma term, called the Christoffel connection, is computed from ordinary derivatives of the metric. This should tell us that Γ^a_{cb} is itself not strictly a tensor in the sense of Eqs. (3.4) and (3.5). It is simply the object that makes the ordinary derivative a tensor as we would like.

The metric, which as I have said will in some sense replace the Newtonian gravitational field in GR, is apparently useful for computing derivatives as well. In GR, spacetime is a differentiable 4 manifold, which will not in general be Euclidean. The metric will be necessary in *any* sort of calculation on this curved manifold, as the

metric encodes all the information about distances on the manifold. The metric on this 4 manifold is a rank 2 tensor, and as such can be written as a 4×4 matrix:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

The metric in general is symmetric, so $g_{\mu\nu} = g_{\nu\mu}$. The inverse of the metric $g_{\mu\nu}$ is $g^{\mu\nu}$, which has the nice property

$$g_{\mu\gamma} g^{\gamma\nu} = \delta^{\nu}{}_{\mu}, \tag{3.7}$$

where $\delta^{\nu}{}_{\mu}$ is the Kronecker delta. This gives us the ability to raise or lower indices of other tensors using the metric. For example,

$$T_{\mu\nu} = g_{\mu\alpha} \, g_{\nu\beta} \, T^{\alpha\beta}$$

and

$$T^{\mu\nu} = g^{\mu\alpha} \, g^{\nu\beta} \, T_{\alpha\beta}.$$

This is why we don't really care whether an index is covariant or contravariant; we can easily switch between the two using the metric. We call this operation contraction with the metric. Tensors can also be contracted with themselves. For example, the Ricci scalar

$$R = R^{\mu}{}_{\mu} \tag{3.8}$$

is formed by contracting the rank 2 Ricci tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}, \tag{3.9}$$

which is itself formed by contracting over the first and third indices of the rank 4 Riemann tensor. The Riemann curvature tensor, as it's known, is a general expression of the intrinsic curvature of a manifold. It is computed from the Christoffel symbols,

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which themselves are computed from the metric. Tensors in GR often come back to the metric or the energy-momentum tensor, which we'll meet momentarily.

We are now in a position to write down and describe the features of the field equations in GR. Just as Maxwell's equations in EM uniquely describe the $\vec{E}$ and $\vec{B}$ fields for a given charge and current density, Einstein's equations uniquely describe the metric $g_{\mu\nu}$ for a given energy-momentum tensor $T_{\mu\nu}$, which contains all the information about the mass and energy distributions. This energy-momentum tensor is the source term in Einstein's equations, just as the charge and current densities are the source terms in Maxwell's equations. The full Einstein field equations (EFE) are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \qquad (3.10)$$

and are referred to in the plural because, as a rank 2 tensor equation, this is actually sixteen equations written in a very compact notation. We endeavor to solve this for the metric $g_{\mu\nu}$. From the Ricci tensor and Ricci scalar involving derivatives of the metric, this turns out to be a set of ten coupled nonlinear partial differential equations; solving the EFE with a given $T_{\mu\nu}$ is no easy task, and often can only be done numerically. We will consider mostly nice situations, where symmetry or some other argument will allow us to simplify the form of the metric.

The first metric found to solve the EFE is for the situation of $T_{\mu\nu} = \mathbf{0}$: an empty universe completely devoid of matter and energy. This metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \eta_{\mu\nu} \qquad \text{Minkowski} \qquad (3.11)$$

is called the Minkowski metric, and corresponds to flat, Euclidean spacetime. This metric turns out to be more useful than in just describing an empty universe. The

next most useful solution to the EFE is the Schwarzschild metric,

$$g_{\mu\nu} = \begin{pmatrix} -(1 - \frac{2M}{r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2M}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2(\theta) \end{pmatrix} \qquad \text{Schwarzschild} \qquad (3.12)$$

which is the metric outside of a non-rotating, uniformly massive sphere. In the Newtonian picture, the gravitational field outside a massive sphere doesn't depend on its radius, just where its center of mass is. Similarly in GR (by Birkhoff's theorem), the metric outside of any non-rotating spherical mass distribution is the Schwarzschild metric [7]. This implies that changing the radius of the sphere only has the effect of changing where the boundary of the sphere is, and has no effect on spacetime outside of that boundary.

Since we're interested in what happens to a gravitational wave incident on a massive boundary, we will have occasion to consider hypersurfaces in spacetime. The hypersurfaces we will consider will be non-null and smooth with well-defined unit normal and tangent vectors. The hypersurface Σ is defined parametrically by $x^\mu = x^\mu(y^a)$, where y^a are the coordinates intrinsic to Σ. Since Σ forms a boundary, we define the region on one side $\mathcal{V}^-$ and the other $\mathcal{V}^+$ and tensors defined in these regions will be denoted with $+$ or $-$ similarly. The spacetime coordinates in $\mathcal{V}^-$ and $\mathcal{V}^+$ are not necessarily the same, but the spacetime coordinates x^μ are defined in some region surrounding Σ where x^μ is the same on both sides of the hypersurface. The tangent curves contained in Σ are

$$e_a^\mu = \frac{\partial x^\mu}{\partial y^a} \qquad (3.13)$$

and the induced metric, or first fundamental form of Σ, is

$$h_{ab}^\pm = g_{\mu\nu}^\pm e_a^\mu e_a^\nu. \qquad (3.14)$$

We let l be the proper distance along geodesics and $l > 0$ in $\mathcal{V}^+$, $l < 0$ in $\mathcal{V}^-$, and $l = 0$ at Σ. Then the unit normal vector to Σ, defined to point from $\mathcal{V}^-$ to $\mathcal{V}^+$, is

$$n_\mu = \epsilon \partial_\mu l \qquad (3.15)$$

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where $n^\alpha n_\alpha = \epsilon \equiv \pm 1$. The extrinsic curvature, or second fundamental form of Σ, is defined as

$$K_{ab} \equiv n_{\mu;\nu} \; e_a^\mu \; e_b^\nu \qquad (3.16)$$

the 3 tensor formed by contracting the covariant derivative of the unit normal vector with the tangent curves in Σ.

4 And (perhaps) in GR

Now that we have developed most of the necessary tools, we follow the procedure laid out in Sect. (2) to attempt to determine the behavior of GWs at an interface. We first derive the wave equation from the EFE. To do this, we again make the simplification of being in a vacuum, with the additional (necessary) simplification of linearizing the EFE. We then discuss boundary conditions, using the tools in Sect. (3.2) to derive general conditions on a non-null hypersurface. We then discuss a gravitational index of refraction. After considering the refractive effect of induced quadrapole radiation and of the background curvature, we will simply posit an index of refraction n.

4.1 Waves in vacuum

To convince ourselves that gravitational waves do exist, we would like to derive the wave equation from the field equations, as in Sect. (2.1). We will have to linearize the EFE to do this in GR. That is, we assume that the metric is the Minkowski metric plus a linear perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon \; h_{\mu\nu} \tag{4.1}$$

where ϵ is a small dimensionless parameter of the metric perturbation $h_{\mu\nu}$. When we plug this into Eq. (3.10), we will discard terms of order ϵ^2 or higher to obtain the so-called linearized EFE. In this linear approximation, we require that the total metric $g_{\mu\nu}$ limit to the Minkowski metric far from any sources (or $\lim_{r\to\infty} h_{\mu\nu} = 0$), and we use $\eta_{\mu\nu}$, the Minkowski metric, to raise and lower indices.

In vacuum, the source term vanishes: $T_{\mu\nu} = 0$. To avoid computing all the necessary Christoffel symbols, I will quote the Riemann tensor as

$$R_{\mu\nu\pi\rho} = \frac{1}{2}\epsilon(h_{\mu\rho,\nu\pi} + h_{\nu\pi,\mu\rho} - h_{\mu\pi,\nu\rho} - h_{\nu\rho,\mu\pi}) + \mathcal{O}(\epsilon^2) \tag{4.2}$$

where we discard the higher order ϵ terms [7]. If this looks like a huge mess, notice the permutations of indices; the Riemann tensor ends up being a simple linear combination of all the ordinary derivatives of the metric perturbation. To get the Ricci

tensor, we contract over the first and third indices, so

$$R_{\mu\nu} = \eta^{\pi\rho} R_{\pi\mu\rho\nu} = \frac{1}{2}\epsilon(h^{\pi}{}_{\mu,\nu\pi} + h^{\pi}{}_{\nu,\mu\pi} - \eta^{\pi\rho}h_{\mu\nu,\pi\rho} - h^{\pi}{}_{\pi,\mu\nu}). \tag{4.3}$$

Note that the third term is really

$$\eta^{\pi\rho}h_{\mu\nu,\pi\rho} = \frac{\partial^2}{\partial t^2}h_{\mu\nu} - \nabla^2 h_{\mu\nu} \equiv \Box h_{\mu\nu}.$$

Contracting again to get the Ricci scalar,

$$R = \epsilon(h^{\mu\nu}{}_{,\mu\nu} + \Box h^{\mu}{}_{\mu}). \tag{4.4}$$

Plugging Eqs. (4.3) and (4.4) into Eq. (3.10) for the EFE (with $T_{\mu\nu} = 0$), we get

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{1}{2}\epsilon(h^{\pi}{}_{\mu,\nu\pi} + h^{\pi}{}_{\nu,\mu\pi} - \Box h_{\mu\nu} - h^{\pi}{}_{\pi,\mu\nu}) - \frac{1}{2}(\eta_{\mu\nu} + \epsilon h_{\mu\nu})\epsilon(h^{\pi\rho}{}_{,\pi\rho} + \Box h^{\pi}{}_{\pi}) \\
&= \frac{1}{2}\epsilon(h^{\pi}{}_{\mu,\nu\pi} + h^{\pi}{}_{\nu,\mu\pi} - \Box h_{\mu\nu} - h^{\pi}{}_{\pi,\mu\nu} - \eta_{\mu\nu}h^{\pi\rho}{}_{,\pi\rho} + \eta_{\mu\nu}\Box h^{\pi}{}_{\pi}) \\
&= 0
\end{aligned} \tag{4.5}$$

discarding terms of $\mathcal{O}(\epsilon^2)$. Equation (4.5) is the linearized form of Einstein's field equations. This is indeed kind of a mess. However, thanks to the gauge-invariance of $R_{\mu\nu}$ and R [6, 7], we can affect the change of variables

$$\psi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\mu}{}_{\mu}. \tag{4.6}$$

Then Eq. (4.5) becomes

$$\frac{1}{2}\epsilon(\psi^{\pi}{}_{\mu,\nu\pi} + \psi^{\pi}{}_{\nu,\mu\pi} - \Box\psi_{\mu\nu} - \eta_{\mu\nu}\psi^{\pi\rho}{}_{\pi\rho}) = 0, \tag{4.7}$$

down to four terms instead of six. Exercising our gauge choice [6, 7], we impose the condition

$$\psi^{\mu}{}_{\nu,\mu} = h^{\mu}{}_{\nu,\mu} - \frac{1}{2}h^{\mu}{}_{\mu,\nu} = 0, \tag{4.8}$$

which is historically called the Einstein gauge. Then (in vacuum), we have

$$\Box\psi_{\mu\nu} = 0$$

which with Eq. (4.8) implies

$$\Box h_{\mu\nu} = 0. \tag{4.9}$$

By linearizing the EFE and imposing Einstein gauge, the vacuum field equations reduce to the wave equation with speed c. Note that this isn't a wave equation in the metric exactly, but in the perturbation to the metric, the existence of which does not necessarily imply the existence of a gravitational field [7]. However, if Eq. (4.9) is true, then by Eq. (4.2),

$$\Box R_{\mu\nu\pi\rho} = 0. \tag{4.10}$$

Given a non-zero Riemann tensor, the field equations guarantee the existence of a gravitational field, and hence perturbations that propagate at speed c.

We will now examine plane gravitational waves. It would be nice if a plane GW could be written like a plane EM wave, as a polarization, an amplitude and a complex exponential containing the wavelength, speed, and phase shift. Without deriving them, let's look at possible plane wave solutions. We can write the metric perturbation of a plane wave propagating in the z direction as

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -h_{22} & h_{12} & 0 \\ 0 & h_{12} & h_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.11}$$

where $h_{22} = h_{22}(k_\mu x^\mu)$ and $h_{12} = h_{12}(k_\mu x^\mu)$. So in fact, all the information about the plane wave is contained in this metric perturbation, with terms that depend on position and a four-dimensional wave vector. This gives us the two polarization states: if $h_{22} = 0$ the wave is called "×-polarized," and if $h_{12} = 0$ it is "+ -polarized." Any plane wave can be written as a superposition of these two polarization states. If we want, we could pull a complex exponential out of this $h_{\mu\nu}$ and write it as

$$h_{\mu\nu} = P_{\mu\nu} e^{ik_\pi x^\pi}. \tag{4.12}$$

Explicitly, for a +-polarized GW with $k_\mu = (\omega, 0, 0, k_z)$ (i.e. traveling in the z direction), with amplitude P_{22},

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -P_{22} & 0 & 0 \\ 0 & 0 & P_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(k_z z - \omega t)}.$$

Keep in mind that in the linear approximation the full metric is still $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$. This wave is a perturbation in Minkowski spacetime. That said, we can basically treat them separately, with the perturbation as our plane wave and Minkowski as the background metric.

4.2   Boundary conditions

Now that we have an idea about gravitational waves, having seen the wave equation derived from the linearized field equations and broadly discussed plane waves, following Sect. (2) the next step in the procedure is to use the field equations to determine boundary conditions. There is a general sense throughout the literature of gravitational physics that the metric itself must be continuous everywhere. Various other boundary conditions are stated with varying levels of rigorous proof.

Just looking at Newtonian gravity, as we saw before we can write the divergence of the Newtonian gravitational potential in direct analogy to the divergence of the electric field as

$$\nabla \cdot \vec{g} = -4\pi G \rho.$$

If we integrate over a Gaussian surface symmetric across a boundary between two regions, with top and bottom area A
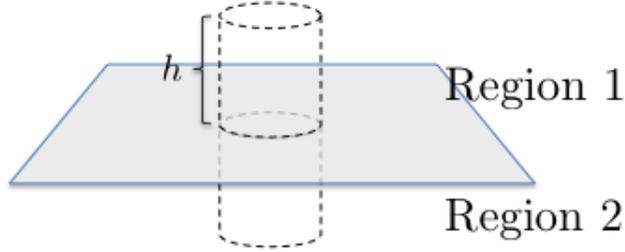
Figure 4: Gaussian surface symmetric across a boundary between two regions.

and again shrink the height h down, we get

$$
\iint \vec{g} \cdot d\vec{a} = -\, 4\pi G \iiint \rho dV
$$
$$
A(g_1^\perp - g_2^\perp) = -\, 4\pi G A \rho
$$
$$
g_2^\perp - g_1^\perp = 4\pi G \sigma \tag{4.13}
$$

which says that the Newtonian gravitational field is discontinuous at a boundary with surface mass density σ. It has been asserted that because the metric in GR must be continuous everywhere, for the Newtonian approximation to agree with this we cannot admit surface mass densities. Kumar [8] describes several others' boundary conditions, which include continuity of the metric, continuity of the first derivative of the metric, continuity of mass density, and continuity in the quantity $g_{a\mu}T^a{}_\nu - g_{a\nu}T^a{}_\mu$. Kumar in particular relaxes the condition of continuity in the first derivative of the metric, which results in δ-function terms entering into the field equations and seems to imply surface mass densities at the discontinuity.

We will try to pare down these boundary conditions. We will conclude with other authors that continuity in the metric must be satisfied at a boundary. Considering the energy-momentum tensor on the hypersurface that defines a boundary, we find another boundary condition in the continuity in the extrinsic curvature of the hypersurface. Allowing for a different metric on either side of the hypersurface will imply an energy-momentum tensor with a singularity at the hypersurface, which is nicely

interpreted as a surface mass density. That is, we may admit surface mass densities without implying discontinuity in the metric.

Following the derivation of boundary conditions from Poisson's *Relativist's Toolkit* [9], we consider a hypersurface Σ with regions $\mathcal{V}^+$ on one side and and $\mathcal{V}^-$ on the other. The hypersurface Σ is defined parametrically by $x^\mu = x^\mu(y^a)$, where the coordinates y^a are intrinsic to Σ and x^μ are the four-dimensional coordinates around the hypersurface, as in Sect. (3.2). Both are the same on either side of Σ. Define the notation for the "jump" across Σ,

$$[A] \equiv A(\mathcal{V}^+)|_\Sigma - A(\mathcal{V}^-)|_\Sigma$$

of a tensor defined piecewise over all $\mathcal{V}^+ \cup \mathcal{V}^-$. Note that the jump of the unit normal vector and the tangent curves on Σ are zero (we defined the unit normal to point from $\mathcal{V}^-$ to $\mathcal{V}^+$). The metric is different in $\mathcal{V}^+$ and $\mathcal{V}^-$, so it is piecewise defined, but we can express the metric everywhere with one equation using the Heaviside step function Θ, defined such that

$$\Theta(l) = \begin{cases} 1 & \text{if } l > 0 \\ 0 & \text{if } l \leq 0 \end{cases}.$$

Then the metric is defined everywhere by

$$g_{\mu\nu} = \Theta(l) \ g_{\mu\nu}^+ + \Theta(-l) \ g_{\mu\nu}^- \tag{4.14}$$

in coordinates x^μ. The first task is to determine whether such a metric is even a valid solution to the EFE. Note that $\partial\Theta(l)/\partial l = \delta(l)$, where $\delta(l)$ is the Dirac delta function, and by the above definition the function $\Theta(l)\delta(l)$ is not well-defined at $l = 0$. So if any terms in the field equations end up containing this fatal combination, such a distribution-valued metric as such is not well-defined. Using the definition of the unit normal vector in Eq. (3.15) and remembering that its jump is zero, the ordinary derivative of the metric is

$$\begin{aligned} \partial_\pi g_{\mu\nu} &= \Theta(l) \ g_{\mu\nu,\pi}^+ + \Theta(-l) \ g_{\mu\nu,\pi}^- + \delta(l)g_{\mu\nu}^+|_\Sigma \epsilon n_\pi - \delta(l)g_{\mu\nu}^-|_\Sigma \epsilon n_\pi \\ &= \Theta(l) \ g_{\mu\nu,\pi}^+ + \Theta(-l) \ g_{\mu\nu,\pi}^- + \epsilon\delta(l)[g_{\mu\nu}]n_\pi \end{aligned} \tag{4.15}$$

so any product of these, such as in the Christoffel symbols (and hence the Riemann tensor, etc.) has terms in $\Theta(l)\delta(l)$. To get rid of these undefined terms, but still allow for such piecewise-define metrics, we require that

$$[g_{\mu\nu}] = 0, \tag{4.16}$$

which is a statement in the particular coordinate system x^μ. Using Eq. (3.14) for the induced metric, and remembering that the jump of the tangent curves on Σ is zero, we can multiply Eq. (4.16) on the right by the tangent curves to get

$$0 = [g_{\mu\nu}]e_a^\mu e_b^\nu = [g_{\mu\nu}e_a^\mu e_b^\nu] = [h_{ab}] \tag{4.17}$$

which is a statement in the coordinates y^a intrinsic to Σ, independent of the coordinates around Σ. This is our first boundary condition: *the induced metric on Σ must be the same on both sides of Σ.*

The second boundary condition comes from considering the energy-momentum tensor at the hypersurface. Reasoning that the Christoffel symbols above and below Σ will be different (thus can be expressed together using the Θ-function), the Riemann tensor, with derivatives of Christoffel symbols, has δ-function terms. We haven't yet specified $g_{\mu\nu}^\pm$, so we don't yet have explicit Christoffel symbols or Riemann tensors in the regions $\mathcal{V}^+$ and $\mathcal{V}^-$, but we can write down the Riemann tensor

$$R^\mu{}_{\nu\pi\rho} = \Theta(l)\ R^{\mu+}{}_{\nu\pi\rho} + \Theta(-l)\ R^{\mu-}{}_{\nu\pi\rho} + \delta(l)A^\mu{}_{\nu\pi\rho} \tag{4.18}$$

as its parts in $\mathcal{V}^+$ and $\mathcal{V}^-$ and its δ-function part at Σ, where

$$A^\mu{}_{\nu\pi\rho} = \epsilon\left([\Gamma^\mu{}_{\nu\rho}]n_\pi - [\Gamma^\mu{}_{\nu\pi}]n_\rho\right). \tag{4.19}$$

It follows that the Ricci tensor and Ricci scalar have δ-function terms, and then that the energy-momentum tensor must as well. Taking $A_{\mu\nu} \equiv A^\pi{}_{\mu\pi\nu}$ and $A \equiv A^\mu{}_\mu$, the energy-momentum tensor is

$$T_{\mu\nu} = \Theta(l)\ T_{\mu\nu}^+ + \Theta(-l)\ T_{\mu\nu}^- + \frac{\delta(l)}{8\pi}(A_{\mu\nu} - \frac{1}{2}Ag_{\mu\nu}) \tag{4.20}$$

[9]. Examining the δ-function part of this, let $8\pi S_{\mu\nu} = A_{\mu\nu} - \frac{1}{2}Ag_{\mu\nu}$. This tensor $S_{\mu\nu}$ is tangent to Σ (i.e., $S_{\mu\nu}n^\nu = 0$), so we can write it as

$$S^{\mu\nu} = S^{ab}e_a^\mu e_b^\nu \tag{4.21}$$

and call S_{ab} the surface energy-momentum tensor. With a bit of work, we would find

$$S_{ab} = -\frac{\epsilon}{8\pi}\Big([K_{ab}] - [K]h_{ab}\Big), \tag{4.22}$$

where K_{ab} is extrinsic curvature and $K \equiv K^a_{\ a}$, but I'd rather just cite Poisson than get into that [9]. If we didn't want a singularity in our energy-momentum tensor, we could just enforce $[K_{ab}] = 0$ as our boundary condition. In general, however, using Eq. (3.16) and remembering that the jump of the tangent curves on Σ is zero,

$$[K_{ab}] = [n_{\mu;\nu}]e^\mu_a e^\nu_b. \tag{4.23}$$

This is our second boundary condition: *the extrinsic curvature is discontinuous at Σ, with the discontinuity given by the jump of the covariant derivative of the unit normal vector.* If we allow for this singularity in the total energy-momentum tensor, the δ-function term has a nice interpretation as a surface mass density. If we choose not to admit such singularities (though as we have seen, they work just fine), then we do not admit surface mass densities. Equations (4.17) and (4.23) are the boundary conditions that will be used henceforth.

4.3   Gravitational refractive index

Now that we've settled on what boundary conditions to use, if we expect to use them as we did in EM we should similarly discuss a gravitational index of refraction. Unlike in optics, we don't have a means to directly experimentally determine a gravitational index of refraction. If we try to compute one exactly as we did for EM waves – from induced dipole radiation in an interface – we would run into the problem that gravitational radiation requires at least a non-zero quadrupole moment. The noninteracting simple harmonic oscillator model of the electrons in an interface really is a zero*th*-order approximation, and we could expect to find a more precise approximation which includes higher order multipole terms including a non-zero quadrupole moment. In particular, Peter Szekeres calculated an index of refraction for GWs propagating through matter composed of particles in which the GW induces a quadrupole moment in each particle [10]. However, unlike the EM case, there is another refractive effect which is more dominant than this induced radiation effect.

P.C. Peters calculated the metric contribution to a gravitational index of refraction, which is significantly larger than previous calculations of a gravitational refractive index from other effects [11]. We will look at this derivation, but in the end, we simply posit a gravitational refractive index n.

We make a linear approximation similar to that in Sect. (4.1) but instead of perturbing away from the Minkowski metric, here we perturb away from the Schwarzschild metric $g_{\mu\nu}^{(0)}$,

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}. \tag{4.24}$$

Expanding the EFE to first order in $h_{\mu\nu}$ gives a wave equation for the spatial components of this perturbation at a fixed time t,

$$(\nabla^2 + \omega^2)\tilde{h}_{ab} = 4\omega^2 \phi \tilde{h}_{ab} \tag{4.25}$$

[11] where

$$\phi = -G\frac{M}{r} \tag{4.26}$$

is the Newtonian gravitational potential for a Schwarzschild-like sphere of mass M and $\tilde{h}_{\mu\nu}$ is defined as

$$\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}^{(0)}g^{(0)\pi\rho}h_{\pi\rho} \tag{4.27}$$

to simplify Eq. (4.25). As in Eq. (4.12) we can write the perturbation as

$$\tilde{h}_{ab} = P_{ab}\tilde{\psi} = P_{ab}e^{ik_a x^a} \tag{4.28}$$

with a constant polarization tensor [11]. Using this, we can now consider a plate of scattering particles in a vacuum at $z = 0$ with thickness Δz, as in the EM case in Sect. (2.4). Suppose a plane GW traveling in the z direction passes through this plate. We similarly write that for small Δz, the wave after the plate is

$$\begin{aligned}
\tilde{\psi}_{after\ plate} &= e^{ik_z z} + \frac{i\omega(n-1)\Delta z}{c}e^{ik_z z} \\
&= \tilde{\psi}_{incident} + \tilde{\psi}_{induced}
\end{aligned} \tag{4.29}$$

the superposition of the incident wave and the induced wave, at a fixed time t which we take to be $t = 0$. The scalar part of solutions to Eq. (4.25) is

$$
\begin{aligned}
\tilde{\psi} &= e^{ik_z z}\left(1 - 2Gmi\omega\left[\ln k_z(r - z) + \int_{k_z(r-z)}^{\infty}\frac{e^{iu}}{u}du\right]\right) \\
&= e^{ik_z z}\left(1 - 2Gmi\omega\int_{k_z(r-z)}^{\infty}\frac{e^{iu}}{u}du\right)
\end{aligned}
\tag{4.30}
$$

and we discard the nonphysical logarithmic phase factor to compute the index of refraction [11]. Normalizing and evaluating the integral, Peters finds

$$
\tilde{\psi} = e^{ik_z z}\left(1 + 4\pi G\rho z\Delta z + \frac{2\pi G\rho\ i\Delta z}{\omega}\right)
\tag{4.31}
$$

for the scalar part of the total wave [11]. The total wave is still $h_{ab} = P_{ab}\tilde{\psi}$. The last term in Eq. (4.31) is the scattered wave we want to attribute to an index of refraction. Comparing Eqs. (4.31) and (4.29), these terms can only be equal if

$$
i\omega(n - 1)\Delta z = \frac{2\pi G\rho\ i\Delta z}{\omega}
$$

or

$$
n = 1 + \frac{2\pi G\rho}{\omega^2}.
\tag{4.32}
$$

So the index of refraction only depends on the gravitational constant, mass density of the plate, and frequency of the wave. Looking at the behavior of this index of refraction, we see that it goes to unity for vanishing mass density (in vacuum) as expected. Comparing to the EM index of refraction Eq. (2.31), there is no ω_0^2 term in the denominator, just the wave frequency ω^2. This is expected, as this method of calculation did not take into account oscillations of massive particles in the plate, so their natural frequencies never came into it. Perhaps most significantly, the fact that n is inversely proportional to the squared frequency of the wave indicates that, like EM waves, matter is dispersive to gravitational waves; matter splits the wave by frequency.

5 Gravitational wave refraction

All that's left is to derive "Fresnel's equations" for a GW. Trying to compute the metric of a planar interface – expecting to then perturb it with a GW – by the Weyl method is shown to be a useless endeavor. Not worrying about the total metric but just looking at how the wave might refract and reflect at an interface, naively trying to follow EM as closely as possible fails, and this attempt ends in a contradiction. We briefly examine the results of Ingraham (working in the Campbell-Morgan formulation of linearized GR) who, upon arriving at the same contradiction, proceeds with new boundary conditions, modified so as not to have such problems. He arrives at Fresnel equations for GWs passing between media with different indices of refraction, in complete analogy with EM.

Not convinced by Ingraham, we attempt to solve the problem again in the usual Einstein formulation in a situation where the metric on both sides of the interface is known. Using the methods of Sect. (3.2) and further, we compute the induced metric on a spherical shell. Our goal then is to apply the boundary conditions and try to get gravitational "Fresnel equations" for a GW incident on this spherical shell.

5.1 First attempts

We want to follow what we did in EM and send a GW through a planar interface. However, in GR a GW is a perturbation in the metric. So, we need to know the background metric for a GW to be meaningful.

Metric of a planar interface

A procedure by which the Schwarzschild metric is easily calculated is called the "Weyl method" (after Hermann Weyl), in which some symmetry of the configuration is exploited to make a simplified metric ansatz, then uses the Einstein-Hilbert form of the action integral to determine its components. If one assumes spherical symmetry and applies the Weyl method, the Schwarzschild metric falls out. Specifically, spherical symmetry means that the metric only deviates from Minkowski in the temporal and radial components, and these deviations are functions of only the radial coordinate.

If we have a planar interface, approximately infinite in the x-y plane, our ansatz is a metric that only deviates from Minkowski in the temporal and z components, as functions of the z coordinate. That is,

$$g_{\mu\nu} = \begin{pmatrix} -A(z) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & B(z) \end{pmatrix}.$$

Without loss of generality, we let $A(z) = a(z)b(z)^2$ and $B(z) = a(z)^{-1}$, in order to get a simple form of the Ricci scalar and the metric determinant. Then our final metric ansatz is

$$g_{\mu\nu} = \begin{pmatrix} -a(z)b(z)^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a(z)^{-1} \end{pmatrix}. \tag{5.1}$$

The Einstein-Hilbert action is

$$S_{EH} = \int d\tau \sqrt{-\det[g_{\mu\nu}]}\, R \tag{5.2}$$

integrating over four-dimensional spacetime. Suppressing the z dependence, the Ricci scalar of this metric is

$$R = \frac{-(3a'b' + ba'' + 2ab'')}{b} \tag{5.3}$$

so the action integral is

$$S_{EH} = \int d\tau \sqrt{-(-b^2)}\, \frac{-(3a'b' + ba'' + 2ab'')}{b} \tag{5.4}$$

$$= \int -(3a'b' + ba'' + 2ab'')dt\,dx\,dy\,dz. \tag{5.5}$$

The x, y, and t integrals can be evaluated to get a real constant in front of the z integral, but this constant will drop out of the next step so we can ignore it. The integrand is then really a Lagrangian in the parameter z,

$$\mathcal{L}(a, a', a'', b, b', b'') = -(3a'b' + ba'' + 2ab'')$$

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which ought to be solveable for the functions $a(z)$ and $b(z)$ using the Euler-Lagrange (E-L) equations,

$$\frac{d^2}{dz^2}\frac{\partial \mathcal{L}}{\partial a} - \frac{d}{dz}\frac{\partial \mathcal{L}}{\partial a'} + \frac{\partial \mathcal{L}}{\partial a''} = 0$$

(similar for b). Plugging this Lagrangian into the E-L equations, we get

$$\frac{d^2}{dz^2}(-b) - \frac{d}{dz}(-3b') + (-2b'') = -b'' + 3b'' - 2b'' = 0 \tag{5.6}$$

and

$$\frac{d^2}{dz^2}(-2a) - \frac{d}{dz}(-3a') + (-a'') = -2a'' + 3a'' - a'' = 0, \tag{5.7}$$

both of which tell us that $0 = 0$, not exactly new information. So, at least using the Weyl method, we can't know the metric outside of a planar interface.

GW at a planar interface

Well, we've come too far to turn back. We'll proceed with what we wanted to do, we just won't know what the background metric is; we might as well assume a flat background as a fallback approximation. As in EM, we posit a planar interface between two media of differing indices of refraction. For simplicity and applicability, we take the first medium to be vacuum. For a plane GW incident on this interface, we assume there exists a reflected wave and a transmitted wave,

$$\begin{aligned} \text{incident} \qquad & h_{\mu\nu} = P_{\mu\nu}e^{i(k_a r^a - \omega t)} \\ \text{reflected} \qquad & h'_{\mu\nu} = P'_{\mu\nu}e^{i(k'_a r^a - \omega' t)} \\ \text{transmitted} \qquad & h''_{\mu\nu} = P''_{\mu\nu}e^{i(k''_a r^a - \omega'' t)}. \end{aligned} \tag{5.8}$$
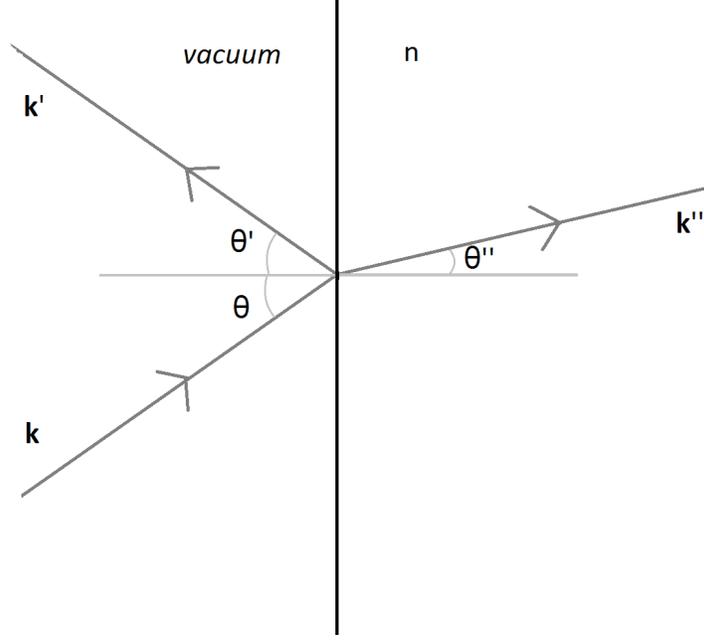
Figure 5: Incident, reflected, and transmitted GWs at an interface.

We immediately know that $k'_a = k_a$ because the incident and reflected waves are both in the vacuum. We make the assumption that, as in EM, the wave speed in the medium is different (probably slower) than the wave speed in vacuum. The only way to account for this is by allowing that $k''_a = nk_a$ for some scalar n, which is what we call the index of refraction. We will also assume that the law of reflection holds, $\theta = \theta'$. Choosing our coordinates so the interface is the x-y plane, we can write each polarization tensor as some constant polarization rotated by the appropriate angle

$$
P_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -f & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}
$$

$$
= f \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ 0 & 0 & \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix} \tag{5.9}
$$

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where the constant amplitude f of the wave is implicit in the polarization tensor. Similarly the wave vectors can be written as a vector whose magnitude is the wave number, rotated by the appropriate angle

$$k_a = k \begin{pmatrix} 0 \\ -\sin(\theta) \\ \cos(\theta) \end{pmatrix}. \tag{5.10}$$

If we then apply the boundary condition that the metric at the interface must be continuous, assuming a flat background means the metric perturbation must be continuous at the interface. This gives

$$h_{\mu\nu}|_{z=0} + h'_{\mu\nu}|_{z=0} = h''_{\mu\nu}|_{z=0} \tag{5.11}$$

which must be true for all t, x, and y. Looking at the origin, this implies $\omega = \omega' = \omega''$, as expected. If we add the condition that $t = 0$, we then have

$$P_{\mu\nu} + P'_{\mu\nu} = P''_{\mu\nu} \tag{5.12}$$

and if this is true at one point on the interface, it must be true everywhere on the interface. The xx element of Eq. (5.12) at $t = z = 0$ is

$$f e^{i(k_y y)} + f' e^{i(k_y y)} = f'' e^{i(k''_y y)} \tag{5.13}$$

which implies

$$f + f' = f'' \tag{5.14}$$

and using $k'' = nk$,

$$k\sin(\theta) = k''\sin(\theta'') = nk\sin(\theta''), \tag{5.15}$$

which is Snell's law. That's an encouraging result. The yy element of Eq. (5.12) is

$$f\cos^2(\theta) + f'\cos^2(\theta') = f''\cos^2(\theta'') \tag{5.16}$$

and the zz element is

$$f\sin^2(\theta) + f'\sin^2(\theta') = f''\sin^2(\theta''). \tag{5.17}$$

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Adding Eqs. (5.16) and (5.17) gives $f + f' = f''$, which was already known. Using Eq. (5.14) and the law of reflection, Eq. (5.16) becomes

$$\cos^2(\theta) = \cos^2(\theta'') \tag{5.18}$$

and Eq. (5.17) becomes

$$\sin^2(\theta) = \sin^2(\theta'') = \frac{1}{n^2}\sin^2(\theta) \tag{5.19}$$

using Snell's law. But Eq. (5.18) implies that

$$\theta = \theta'' + \imath\pi \tag{5.20}$$

(where $\imath$ can be 1 or 0), which only agrees with Eqs. (5.15) and (5.19) if $n = \pm 1$, contradicting our assumptions. Therefore this method of directly proceeding as in EM does not work.

In this situation, metric continuity implies that the frequency doesn't change moving between media, Snell's law holds, and that there's a relationship between the amplitudes of the incident, reflected, and transmitted waves. However, the same condition also implies a contradiction, so perhaps none of these implications are valid. If we knew the actual background metric, applying boundary conditions would look a lot more interesting, and we might be able to determine the reflected and transmitted waves, or at least not run into a contradiction.

5.2   Salvation?

Beginning this research, I believed no one had published anything describing gravitational "Fresnel equations," either because this problem is uninteresting or thought to be trivial to an expert in the field. However, in 1997 one R.L. Ingraham published "Gravitational Waves in Matter" [12], in which the author also believes to be the first to tackle this problem. They suggest the reason for this is that it has no immediate application to GW detection. I suspect others probably solved the problem long before Ingraham or I, but for whatever reason I've been unable to find their work. Ingraham works in the Campbell-Morgan (CM) formulation of linearized GR [13] in

which the metric is recast as two dyads that resemble the electric and magnetic field, with field equations that resemble Maxwell's equations. Not wishing to plumb the details of the CM formulation, we examine Ingraham's paper rather qualitatively.

With these $\mathbf{E}_{ij}$ and $\mathbf{B}_{ij}$ dyads, Ingraham treats the problem of $\mathbf{E}_{ij}$ and $\mathbf{B}_{ij}$ waves incident on a planar interface between two effectively infinite media of differing indices of refraction. Using the boundary condition that the $\mathbf{B}_{ij}$ dyad must be continuous across the boundary, Ingraham arrives at a contradiction. In particular, this boundary condition implies (using our previous primed notation for reflected and transmitted quantities)

$$\beta + \beta' = \beta'' \tag{5.21}$$

$$(\beta + \beta') \cos^2(\theta) = \beta'' \cos^2(\theta'') \tag{5.22}$$

which is precisely Eqs. (5.14) and (5.16), the contradiction we arrived at in the previous section by simply treating the wave in $h_{\mu\nu}$ as we did an EM wave.

Ingraham goes on to adopt new boundary conditions so as to avoid this contradiction. The "smoothness principle" adopted hypothesizes

a) components of the dyad fields normal to the interface are discontinuous across the interface,

b) only the traceless part of the tangential components of the dyad fields are smooth across the interface.

Ingraham arrives at "Fresnel equations" for a simplified model with constant susceptibility and no dispersion. Ingraham admits that constant susceptibility may not be physical, though it simplifies calculations. It's hard to be sure of the meaning of some of the paper because of the obscurity of the CM formulation, but the truly opaque assertions are the "smoothness principle" and lack of dispersion. As we saw from the index of refraction calculation, matter is dispersive to GWs. It's unclear to us whether simply replacing Ingraham's index of refraction with a dispersive index of refraction would correct this. That would be an astoundingly simple solution.

5.3 Last efforts

Rather than dig into the CM formulation and Ingraham's paper, we would prefer to continue working in the usual Einstein formulation of GR, and hope the boundary conditions from Poisson will be elucidating. To make the problem tractable, we really need to know the background metric on both sides of the boundary. The metrics we're most familiar with are Minkowski and Schwarzschild, so it would be nice if we could come up with some hypersurface that is spherically symmetric but flat in its interior. In fact, we can; this is just a spherical shell. We can take a spherical shell (say composed of uniform noninteracting matter, or "dust"), and while we expect its radius to change in time, by Birkhoff's theorem the metric outside will always be Schwarzschild and the metric inside will always be Minkowski [7, 9].

The spherical hypersurface Σ is described parametrically by $R(\tau) = r$ and $T(\tau) = t$, where τ is the proper time measured by an observer comoving with the shell at radial coordinate r and temporal coordinate t. The flat region inside the spherical shell will be referred to by $\mathcal{V}^-$, and outside the shell with the Schwarzschild metric will be $\mathcal{V}^+$. The coordinates on Σ are $y^a = (\tau, \theta, \phi)$ and the coordinates surrounding Σ are $x^\mu = (t, r, \theta, \phi)$. In this section, an overdot represents a partial derivative with respect to the proper time τ. Our goal is to compute the induced metric and extrinsic curvature and enforce the boundary conditions on these derived in Sect. (4.2). We first compute the tangent curves on Σ,

$$e_\tau^\mu = \frac{dx^\mu}{d\tau} = (\dot{T}, \dot{R}, 0, 0), \tag{5.23}$$

$$e_\theta^\mu = \frac{dx^\mu}{d\theta} = (0, 0, 1, 0), \tag{5.24}$$

$$e_\phi^\mu = \frac{dx^\mu}{d\phi} = (0, 0, 0, 1). \tag{5.25}$$

The vector normal to all of these and pointing from $\mathcal{V}^-$ to $\mathcal{V}^+$ is $\tilde{n}_\mu = (-\dot{R}, \dot{T}, 0, 0)$. Normalizing this vector,

$$\tilde{n}_\mu g^{\mu\nu} \tilde{n}_\nu = -F^{-1}\dot{R}^2 + F\dot{T}^2 = 1. \tag{5.26}$$

We define

$$F \equiv 1 - \frac{2M}{R} \tag{5.27}$$

$$\beta^+ \equiv \sqrt{\dot{R}^2 + F} = F\dot{T} \tag{5.28}$$

$$\beta^- \equiv \sqrt{\dot{R}^2 + 1}. \tag{5.29}$$

Now looking at the extrinsic curvature, $K_{ab} = n_{\mu;\nu} e_a^\mu e_b^\nu$. The off-diagonal terms trivially vanish because the tangent curves are mutually orthogonal. Computing the diagonal terms of K_{ab}^+ (the $+$ will be suppressed),

$$K_{\theta\theta} = n_{\theta;\theta} = n_{\theta,\theta} - \Gamma^\mu_{\ \theta\theta} n_\mu = \beta_+ R, \tag{5.30}$$

and raising one index with the Schwarzschild metric,

$$K^\theta_{\ \theta} = \frac{\beta_+}{R}. \tag{5.31}$$

The next term is similar:

$$K_{\phi\phi} = n_{\phi;\phi} = -\Gamma^\mu_{\ \phi\phi} n_\mu = \beta_+ R \sin^2(\theta) \tag{5.32}$$

and raising one index gives

$$K^\phi_{\ \phi} = \frac{\beta_+}{R}. \tag{5.33}$$

The $K_{\tau\tau}$ term requires disproportionately more work. Quoting Poisson,

$$K^\tau_{\ \tau} = \frac{\dot{\beta}_+}{\dot{R}} \tag{5.34}$$

[9]. The K_{ab}^- terms are similar:

$$K^\theta_{\ \theta} = K^\phi_{\ \phi} = \frac{\beta_-}{R} \tag{5.35}$$

$$K^\tau_{\ \tau} = \frac{\dot{\beta}_-}{\dot{R}} \tag{5.36}$$

[9]. Discontinuity in the extrinsic curvature implies a surface mass density. The surface energy-momentum is then written as

$$S^{ab} = \sigma \, u^a u^b \tag{5.37}$$

where u^a is the velocity vector of the hypersurface. The induced metric must by our first boundary condition be the same on both sides of Σ, and is

$$h_{ab} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2 \sin^2(\theta) \end{pmatrix}. \tag{5.38}$$

Knowing the metric everywhere, including the induced metric on the surface, along with the two boundary conditions, will hopefully allow the behavior of a GW incident on this hypersurface to be determined. By someone else.

6 Conclusions

We set out to determine how gravitational waves behave when traveling from vacuum into matter. We examined the well-known electromagnetic case first to familiarize ourselves with the procedure we expected to use in the gravitational case. After discussing a gravitational index of refraction and showing the derivation of an index from the dominant refractive effect, we attempted to use the same procedure from the electromagnetic case for gravitational waves at a planar interface. However, without knowledge of the background metric, this naive treatment of gravitational waves at an interface failed, resulting in an intractable contradiction. With no known solution at hand, we reexamined our assumptions and boundary conditions. For a spherical shell of dust, we found the induced metric on this spherical hypersurface and computed the extrinsic curvature. The next step is to perturb this metric with a gravitational wave. Questions still remain, such as whether this perturbing wave affects the induced metric on the spherical hypersurface, and indeed, what the behavior of a gravitational wave at this or any other interface is.

7 Acknowledgements

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