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Actuarial Ruin Theory

Haley Reed

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Actuarial ruin theory utilizes mathematical models to represent an insurer's vulnerability to ruin (where ruin is defined as the financial state of a negative surplus). Both continuous and discrete models, as well as a variety of distribution families are used in ruin theory to describe the risks associated with a group of insurance policies.

This paper begins with a discussion of the broader topic of risk theory. We then will move into the specific motivations for ruin theory before diving into the mathematical theory. Then, time of ruin will be discussed and defined. Subsequently, the Compound Poisson Surplus Model will be defined. Following the definitions presented in this section, we present the Poisson Process and the Compound Surplus Process, with relevant proofs. The probability of eventual ruin, for the Compound Poisson Surplus Model, is also discussed. This paper concludes with a summary and a recommendation for future material for the interested reader.

1 Background

1.1 Risk Theory

Risk theory, as it exists in the study of actuarial mathematics, is interested in modeling claim amounts. It is split between individual risk models and collective risk models. Individual risk models seek to model claim amounts at the individual claim level, and these random variables are generally summed together to get estimates on total portfolio risk. However, it often makes more sense to model risk on a portfolio level. These models are known as collective risk models. And while some amount of policy information is ignored in these models, they are generally "both computationally efficient and rather close to reality" [4].

The formal collective risk model is motivated by the following scenario: we have a fixed period of time and a collection of policies for which we would like to predict S, the total claims amount over that period.

1.2 Ruin Theory

Ruin theory sits within the broader field of risk theory. While traditional risk theory is defined only for a fixed period of time, and is either individual or collective, ruin theory is a dynamic multi-period approach to risk theory focused primarily on the development of $U(t)$, the insurer's capital or surplus over time t [4].

Traditional risk theory was developed by Swedish actuary Ernst Filip Oskar Lundberg in 1907 [1]. It is incredibly helpful in modeling insurer's vulnerability to insolvency/ruin. Ruin theory, and its component models, are necessary for both long-term financial planning and comparing relative benefits and risks of portfolios [4].

2 Time of Ruin

We are particularly interested in the case where surplus, $U(t)$, becomes negative. When this happens we say ruin has occurred. One may be interested in calculating the probability of ruin over some finite time horizon, t. This is an incredibly important concept, as the approximation and "calculation of the probability of ruin is one of the central problems in actuarial science" [4]. The mathematical notion of this probability is provided in the following definition,

Definition 1

Let T be the first time surplus becomes negative. Then,

$$T = min\{t : U(t) < 0\}$$

The probability that ruin will occur before a fixed time t, $\psi(u,t)$, given an initial surplus of u at $t = 0$, is:

$$\psi(u,t) = P(T < t | U_0 = u)$$

While the calculation of $\psi(u,t)$ is possible for relatively simple discrete cases of $U(t)$, they are infeasible for most discrete cases and virtually all continuous cases [5]. Fortunately, the probability of eventual ruin provides a convenient upper bound. The definition for the probability of eventual ruin, $\psi(u)$ is given here,

Definition 2

Let T be the first time surplus becomes negative. The probability that ruin will occur eventually, $\psi(u)$, given an initial surplus of u at $t = 0$, is:

$$\psi(u) = P(T < \infty | U_0 = u)$$

It is worth noting that $\psi(u,t) \leq \psi(u)$ for all t. We will return to the problem of calculating $\psi(u)$ specifically for Compound Poisson Surplus Models later in this paper.

3 Continuous Poisson Surplus Model

3.1 Definitions

While we have explored different topics within ruin theory, and even risk theory as a whole, this paper explores more deeply the Continuous Poisson Surplus Model. Defined by David Promislow [5] to be,

Definition 3

Let $U(t)$ be the surplus amount at time t, let u be the initial surplus amount, let c be the amount of premium continuously collected each period, and let $S(t)$ be the insurer's aggregate claims up to time t. Then, the formula for the Continuous Poisson Surplus Model is given by,

$$U(t) = u + ct - S(t)$$

What differentiates this model from other surplus models is how we define $S(t)$. Its definition is given below.

Definition 4

In this model, $S(t)$ is a Compound Poisson process. Meaning that we can model the insurer's aggregate claims with both a severity distribution, X, and a Poisson frequency distribution, $N(t)$. $S(t)$ is then given by,

$$S(t) = \sum_{k=1}^{N(t)} X_k$$

Where $\{\ X_k\ \}$ are independent, each has the same distrubtion as X, and they are independent of $N(t)$. In mathematical notion,

$$S(t) \sim \Big\langle N(t), X \Big\rangle$$

3.2 Poisson Process

By definition of this distribution, $N(t)$ is a Poisson process. It worth defining and exploring the Poisson process more here,

Definition 5

The Poisson process, $N(t)$, is a counting process with rate λ if it has stationary and independent increments and if, for all $t > 0$, the number of events occurring in the interval [0,t] is given by,

$$N(t) \sim Poisson(\lambda t)$$

Adapting Gallager's definition [2],

Definition 6

For a Poisson process of rate λ, where $t > 0$, the PMF for $N(t)$ (i.e. the number of arrivals in the time interval [0,t]) is given by

$$P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Using David Promislow's definition, we say that the counting process has independent increments if the number of events that occur in disjoint time intervals are independent.

The Poisson process is helpful in modeling events where the number of occurrences in any time interval is a Poisson distribution, with the λ parameter proportional to the length of the interval. As is true with the basic Poisson distributions, these events should be relatively rare in order for the Poisson process to be a wise choice for modeling. We also note the following properties of the Poisson process briefly:

Definition 7

For a Poisson process with rate λ, where $t > 0$,

$$E(N(t)) = \lambda t$$

$$Var(N(t)) = \lambda t$$

Using these properties, a worked example, with questions adapted from *Fundamentals of Actuarial Mathematics* [5], is given below:

In a Poisson process, the probability that exactly one event will occur in any given hours is $3e^{-3}$. *What is the probability that exactly two events will occur in any 20 minute period?*

Before we can answer the question presented, we must find λ. Using Definition 6, $k = 1, P(X = 1) = 3e^{-3}$, and $t = 1$, we can do so.

$$\begin{aligned} P(X = k) &= \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\ P(X = 1) &= \frac{(\lambda \times 1)^1 e^{-\lambda \times 1}}{1!} \\ 3e^{-3} &= \lambda e^{-\lambda} \\ \lambda &= 3 \end{aligned}$$

We can then move forward and answer the question at hand. Using Definition 6 and recognizing that, in this situation, $k = 2$, $\lambda = 3$, and $t = \frac{1}{3}$ because 20 minutes is one third of an hour, and we are interested in the number of arrivals in any given [h,h+20 minutes] period (which is the same as the number of arrivals in [0 minutes, 20 minutes] because we are assuming that increments of over disjoint time intervals are independent).

$$
\begin{aligned}
P(X=2) &= \frac{(3 \times \frac{1}{3})^2 e^{-3\times\frac{1}{3}}}{2!} \\
&= \frac{(1)^2 e^{-1}}{2!} \\
&= \frac{e^{-1}}{2} \\
&\approx 0.1839397
\end{aligned}
$$

The approximate probability that exactly two events occur in any 20 minute period is 18.39%.

For the same Poisson process, if you start observing the process at some point of time, what is the probability that it will be less than 10 minutes until an event occurs?

Given the nature of the PMF of the Poisson process, it is much easier to calculate the complement of this event (the probability of observing no events in that 10 minute period) than the event of interest itself. We compute the complement's probability using Definition 6 because the the probability of 0 arrivals in the [0 minute, 10 minute] interval is the same as the probability in any other interval because we are told this is a Poisson process, implying all increments of time are independent. We can compute the probability of this complement using Definition 6 and recognizing that in the situation of the complement, $k = 0$, $= 3$, and $t = \frac{1}{6}$ because 10 minutes is one sixth of an hour. The calculations are presented here,

$$
\begin{aligned}
P(X=k) &= \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\
P(X=0) &= \frac{(3\times\frac{1}{6})^0 e^{-3\times\frac{1}{6}}}{0!} \\
&= \frac{(\frac{1}{2})^0 e^{-\frac{1}{2}}}{1} \\
&= e^{-\frac{1}{2}} \\
&\approx 0.6065307
\end{aligned}
$$

So, the probability that it will be less than 10 minutes until an event occurs is approximately (1-0.6065307), or 39.35%, by the Complement Rule.

3.3 Defining $S(t)$: the Compound Poisson Process

The compound Poisson process has some interesting standard properties. We explore a couple of those with the a proof of $E[S(t)]$ and $Var[S(t)]$ as follows.

Proposition

For the compound Poisson process, $S(t)$, with rate λ and where $t > 0$,

$$E[S(t)] = \mu\lambda t$$

Proof. By the definition of expected values,

$$E[S(t)] = E\left[\sum_{k=1}^{N(t)} X_k\right]$$

We move forward by using the law of total expectation, which states that if X and Y are two random variables that exist on the same probability space and E(X) is defined, then $E(X) = \sum_i E(X|A_i)P(A_i)$. While the proof of the law of total expectation is beyond the scope of this paper, we can use it here.

$$E[S(t)] = \sum_{n=0}^{\infty} \Big(\big(E[S(t)|N(t) = n] \big) \times Pr(N(t) = n) \Big)$$

The conditional expectation for $S(t)$, given that there have been n events, is n multiplied by $E[X]$, because X_ks are identically and independently distributed random variables. Letting $\mu = E[X]$, $E[S(t)|N(t) = n] = n\mu$. So,

$$E[S(t)] = \sum_{n=0}^{\infty} \Big(n\mu \times Pr(N(t) = n) \Big)$$

Using that the PMF of $N(t)$ is $P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$,

$$E[S(t)] = \sum_{n=0}^{\infty} \left(n\mu \times \frac{(\lambda t)^n e^{-\lambda t}}{n!} \right)$$

Rearranging the n terms and moving some of the constants out front,

$$E[S(t)] = \mu e^{-\lambda t} \lambda t \sum_{n=1}^{\infty} \left(\frac{(\lambda t)^{n-1}}{(n-1)!} \right)$$

Then, subsisting $y = n - 1$,

$$E[S(t)] = \mu e^{-\lambda t} \lambda t \sum_{y=0}^{\infty} \left(\frac{(\lambda t)^y}{y!} \right)$$

We then continue by recognizing that we can use the Maclauren expansion of the exponential equation, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$$E[S(t)] = \mu e^{-\lambda t} \times \lambda t e^{\lambda t}$$

Then, $e^{-\lambda t}$ and $e^{\lambda t}$ multiplied together cancel each other out. So,

$$E[S(t)] = \mu \lambda t$$

□

Proposition

For the compound Poisson process, $S(t)$, with rate λ and where $t > 0$,

$$Var[S(t)] = \lambda t(\sigma^2 + \mu^2)$$

Proof. Using the variance equivalence proposition,

$$Var[S(t)] = E[S(t)^2] - (E[S(t)])^2$$

We will begin by finding $E[S(t)^2]$. By the definition of expected values,

$$E[S(t)^2] = E\left[\left(\sum_{n=1}^{N(t)} X_k\right)^2\right]$$

By the law of total expectation,

$$E[S(t)^2] = \sum_{n=0}^{\infty}\Big(E[S(t)^2|N(t) = n] \times Pr(N(t) = n)\Big)$$

$S(t)^2$ is the sum of n independent and identically distributed random variables, squared. So,

$$\begin{aligned} S(t)^2 &= (X_1 + X_2 + ... + X_n)^2 \\ &= X_1^2 + X_2^2 + X_3^2 + ... + 2X_1X_2 + 2X_1X_3 + ... \end{aligned}$$

So returning to the expected value problem,

$$E[S(t)^2|N(t) = n] \;=\; E(X_1^2) + E(X_2^2) + E(X_3^2) + ... + E(2X_1X_2) + E(2X_1X_3) + ...$$

We then recognize that $E(2X_1X_2) = 2E(X_1)E(X_2)$ because we are told these are independent variables. And because we are told these are identical variables, $E(X) = E(X_1) = E(X_2) = ... = E(X_n)$. So $2E(X_1)E(X_2) = 2E(X)^2$. Additionally, note that because these variables are identical, it also follows that $E(X^2) = E(X_1^2) = E(X_2^2) = ... = E(X_n^2)$. Additionally, since $S(t)^2$ is calculated by squaring the sum of n terms, there are n $E(X^2)$ terms and $\frac{n(n-1)}{2}$ $2E(X)^2$ terms. So,

$$E[S(t)^2|N(t)=n] \quad = \quad nE(X^2)+n(n-1)E(X)^2$$

We also know that $\mu = E[X]$, so we can rewrite this equation slightly:

$$E[S(t)^2|N(t)=n] \quad = \quad nE(X^2)+n(n-1)\mu^2$$

We also know that $Var[X] = \sigma^2 = E(X^2) - \mu^2$. So, $E(X^2) = \sigma^2 + \mu^2$. Therefore,

$$\begin{aligned} E[S(t)^2|N(t)=n] &= n(\sigma^2+\mu^2)+n(n-1)\mu^2 \\ &= n\sigma^2 + n\mu^2 + (n^2-n)\mu^2 \\ &= n\sigma^2 + n\mu^2 + n^2\mu^2 - n\mu^2 \\ &= n\sigma^2 + n^2\mu^2 \end{aligned}$$

Using that the PMF of $N(t)$ is $P(N(t)=n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$,

$$\begin{aligned} E[S(t)^2] &= \sum_{n=0}^{\infty}\left(n\sigma^2 + n^2\mu^2 \times \frac{(\lambda t)^n e^{-\lambda t}}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(n\sigma^2 \times \frac{(\lambda t)^n e^{-\lambda t}}{n!}\right) + \sum_{n=0}^{\infty}\left(n^2\mu^2 \times \frac{(\lambda t)^n e^{-\lambda t}}{n!}\right) \\ &= \sigma^2\sum_{n=0}^{\infty}\left(n \times \frac{(\lambda t)^n e^{-\lambda t}}{n!}\right) + \mu^2\sum_{n=0}^{\infty}\left(n^2 \times \frac{(\lambda t)^n e^{-\lambda t}}{n!}\right) \end{aligned}$$

We then use that $\left(n \times \frac{(\lambda t)^n e^{-\lambda t}}{n!}\right) = E[N(t)]$ and $\left(n^2 \times \frac{(\lambda t)^n e^{-\lambda t}}{n!}\right) = E[N(t)^2]$ to rewrite the equation as follows,

$$E[S(t)^2] \quad = \quad \sigma^2 E[N(t)] + \mu^2 E[N(t)^2]$$

Now, we have a satisfactory equivalence for $E[S(t)^2]$, we can return to solving for variance. Using what we just found, $E[S(t)^2] = \left(\sigma^2 E[N(t)] + \mu^2 E[N(t)^2]\right)$, and the result of the last proof, $E[S(t)] = \mu\lambda t$, we can begin with:

$$\begin{aligned} Var[S(t)] &= E[S(t)^2] - (E[S(t)])^2 \\ &= \sigma^2 E[N(t)] + \mu^2 E[N(t)^2] - (\mu\lambda t)^2 \end{aligned}$$

To better match terms, we use that $\mu\lambda t = \mu E[N(t)]$. So,

$$\begin{aligned} Var[S(t)] &= \sigma^2 E[N(t)] + \mu^2 E[N(t)^2] - (\mu E[N(t)])^2 \\ &= \sigma^2 E[N(t)] + \mu^2 E[N(t)^2] - \mu^2 E[N(t)]^2 \\ &= \sigma^2 E[N(t)] + \mu^2 \Big(E[N(t)^2] - E[N(t)]^2 \Big) \end{aligned}$$

We then notice that $\Big(E[N(t)^2] - E[N(t)]^2 \Big)$ is equal to $Var[N(t)]$. So,

$$Var[S(t)] = \sigma^2 E[N(t)] + \mu^2 Var[N(t)]$$

We then use the following properties of the Poisson process, $E[N(t)] = \lambda t$ and $Var[N(t)] = \lambda t$. So,

$$\begin{aligned} Var[S(t)] &= \sigma^2 \lambda t + \mu^2 \lambda t \\ &= \lambda t \Big(\sigma^2 + \mu^2 \Big) \end{aligned}$$

□

3.4 Graphic Representation of $U(t)$

Now that we have a basic understanding of $S(t)$, and some of its properties, we can better understand the compound Poisson surplus model as a whole. As a reminder, the formula for the continuous Poisson surplus model is $U(t) = U + ct - S(t)$. A graphic representation for a possible realization of this model is given below,

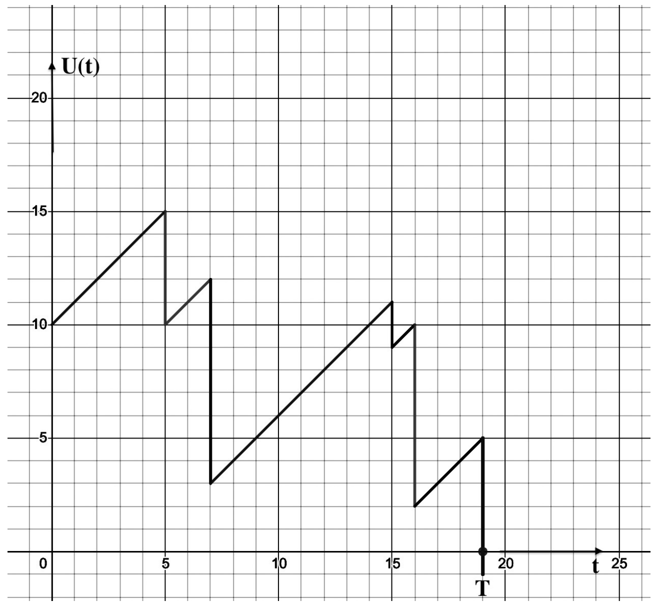

Figure 1: A realization of the continuous-time surplus process, with $u = 10$, $c = 1$, and time of ruin at $t = 19$. As shown graphically, $S(t)$ is random in both severity (as shown by the length of vertical lines) and frequency (as shown by the length of the time (t) interval between each vertical line. Constant premium rate , c, can be shown through the slope of the diagonal lines. Graphic was made with Desmos.

3.5 Probability of Ruin for Compound Poisson Surplus Models

As discussed earlier, we will not attempt to find $\psi(u,t)$ directly. Rather, we can calculate $\psi(u)$, which provides the probability of eventual ruin and provides the upper bound for the probability of ruin at any time, t.

Before discussing the general case, $\psi(u)$, we will investigate $\psi(0)$. For any severity distribution, X, we can calculate $\psi(0)$, the probability of eventual ruin when the initial surplus is 0. While interpreted basically, this is of little interest because virtually no rational actor would start an insurance company with surplus 0, $\psi(0)$ also provides the probability that for any starting value (u), the probability that surplus will eventually be less that u [5]. For the continuous

Poisson surplus model with any severity distrubition X,

$$\psi(0) = \frac{1}{1+\theta}$$

Where θ is the risk load, interpreted in context as the additional fraction of the equivalence principal premium added to the base equivalence principal premium when calculating the total premium. Note the equivalence principal premium equals the present expected value of all future claims. So, total premium $=$ $(1+\theta) \times$ equivalence principal premium.

Definition 8

In the case of the continuous Poisson surplus model, the risk load θ is given by:

$$\theta = \frac{c}{\lambda E(X)} - 1$$

So, we can rewrite $\psi(0)$ as follows,

Definition 9

For a compound Poisson surplus model with rate λ, and expected value of severity distribution $E(X)$, and a premium of c collected each period, where $t > 0$, the probability of eventual ruin where the initial surplus is 0 is given by:

$$\psi(0) = \frac{\lambda E(X)}{c}$$

Now, returning our attention to the general case, we have to first define a new random variable, R.

Definition 10

An adjustment coefficient of a random variable X is a positive number R satisfying

$$M_X(-R) = 1$$

Now, we can define the probability of eventual ruin,

Definition 11

For any compound Poisson surplus model with existing adjustment coefficient R, initial surplus u, and deficient at the time of ruin $D(u)$, the probability of eventual ruin is given by:

$$\psi(u) = \frac{e^{-Ru}}{E(e^{RD(u)})}$$

For the purposes of this paper, we will more carefully explore the case when X is defined by the exponential distribution ($X \sim \text{Exp}(\beta)$). β replaces the typical parametrization of λ to avoid confusion between frequency and rate parameters.

While the proof is beyond the scope of this paper, when $X \sim \text{Exp}(\beta)$, $D(u) \sim \text{Exp}(\beta)$ as well for all $u \geq 0$. We too will leave the proof, but for the exponential case of X, R is given by:

$$R = \frac{\beta\theta}{1+\theta}$$

Recognizing that $E(e^{RD(u)}) = M_{D(u)}(R)$, and $M_{D(u)}(R) = M_X(R)$ when X has an exponential distribution [5],

$$\begin{aligned} E(e^{RD(u)}) &= M_X(R) \\ &= 1+(1+\theta)E(X)R \\ &= 1+(1+\theta)\frac{1}{\beta}\left(\frac{\beta\theta}{1+\theta}\right) \\ &= 1+\frac{1}{\beta}\left(\frac{\beta\theta}{1}\right) \\ &= 1+\theta \end{aligned}$$

This replaces the denominator in the equation of Definition 11. So,

Definition 12

For the exponential case of X, with initial surplus u, risk load θ, and exponential parameter β, the compound surplus model has probability of eventual ruin,

$$\psi(u) = \frac{1}{1+\theta}e^{-u\frac{\beta\theta}{1+\theta}}$$

Then, to provide an example, we can answer a basic question to show the mechanism and the usefulness of this formula.

For an insurer that starts business today with an initial surplus of \$10,000, charging their group of insureds a total of \$2,000 in premiums each month at a constant rate, whose claims can be modeled with a Poisson frequency distribution with an expected value of two claims a month, and whose severity distribution can be modeled with an exponential distribution with expected value \$833.33 per claim, what is the probability of eventual ruin?

We are told that the initial surplus is \$10,000, so $u = 10000$. We are told that the insurer receives \$2,000 in premiums each month, so $c = 2000$. We are told that the expected number of claims per month is 2, and that the frequency can be modeled with a Poisson distribution, so $\lambda = 2$. We are also told that the severity distribution, X, can be modeled with an exponential distribution with $E(X) = 833.33$. We know that for an exponential random variable, X, $E(X) = \frac{1}{\beta}$. So,

$$\frac{1}{\beta} = 833.33$$
$$\beta = .0012$$

We then can calculate θ as follows (using Definition 8),

$$\begin{aligned}
\theta &= \frac{c}{\lambda E(X)} - 1 \\
\theta &= \frac{2000}{2 \times 833.33} - 1 \\
\theta &= .2
\end{aligned}$$

We know have all the parameters to calculate $\psi(u)$ (using Definition 12),

$$\begin{aligned}
\psi(u) &= \frac{1}{1+\theta} e^{-u\frac{\beta\theta}{1+\theta}} \\
\psi(10000) &= \frac{1}{1+.2} e^{-10000(\frac{.0012\times .2}{1+.2})} \\
&= \frac{1}{1.2} e^{-2} \\
&= \frac{1}{1.2} e^{-2} \\
&= 0.1127794
\end{aligned}$$

So the probability of eventual ruin, under this model, is approximately 11.28%.

4 Conclusion

In this paper, we have discussed the background and the applied uses of ruin theory in actuarial science. We also defined time of ruin and the probability of eventual ruin, providing motivation for the rest of the paper. The majority of this paper then focused on the Compound Poisson Surplus Model. The definitions of this specific model, along with the Poisson process, and the compound Poisson process were presented, while drawing the connections between the three. Examples and proofs were provided in discussion of the Poisson process and the compound Poisson process to demonstrate their function and basic properties. Finally, the steps for calculating the probability of eventual ruin for the Compound Poisson Surplus Model with an exponential severity distribution were presented, with an example to demonstrate the applied use of ruin theory in actuarial modeling.

While not covered in this paper, the interested reader could explore the discrete models within ruin theory. One such model is the stopping time model for discrete-time stochastic processes. These models utilize Markov Chain setups to solve problems like optimal stopping times. Material on this subject can be found in section 21.3 in David Promislow's *Fundamentals of Actuarial Mathematics* [5].

The classical Lundberg surplus model has limitations in terms of applications. More modern insurance risk models can be used to find more realistic modeling approaches to ruin theory. One such model, introduced by Hans Gerber and Elias Shui, generalizes the classical risk model by discounting with respect to the the time of ruin to model the surplus immediately before ruin and the deficit at the time of ruin [3]. Exploring the Gerber Shui models would be a good next step in developing knowledge beyond the scope of the classical Lundberg surplus model.

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