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Symmetry Methods and Self-Similar Solutions to Curve Shortening

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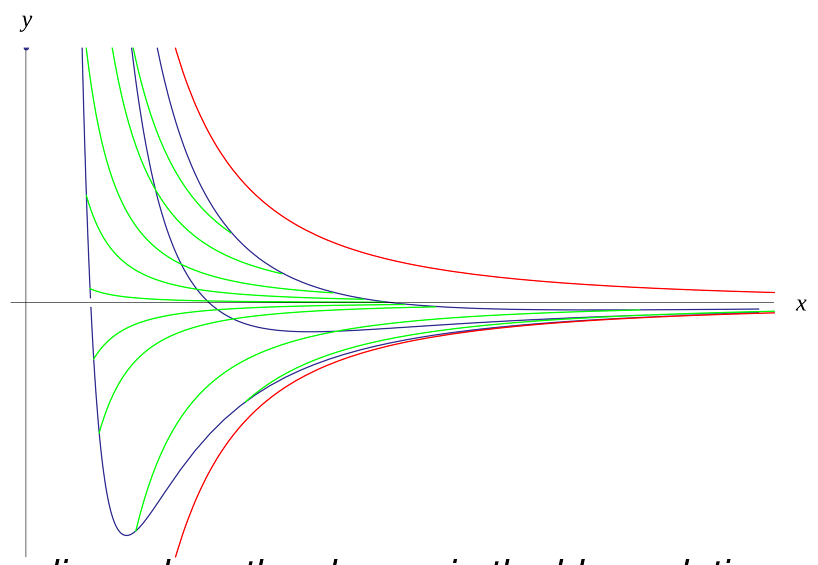
Abstract

Curve shortening is a geometric process that continually evolves a curve based on its curvature. Self-similar solutions to the curve shortening equation maintain their form throughout the process, though they can be scaled, translated, or rotated. These self-similar solutions correspond to the invariant solutions of the symmetry method for solving differential equations.

1. Symmetry methods

- Symmetry methods are a technique for solving differential equations.
- A symmetry for a differential equation maps solutions to solutions, for example by scaling or translating.
- The goal is to use a symmetry to turn the differential equation into a form that is easier to solve by normal methods (e.g., separation of variables)
- Symmetries exist in one-parameter families that produce flows where solutions are continuously mapped to solutions (as the value of the parameter changes).
- Example [3]: The scaling transformation $(\hat{x}, \hat{y}) = (e^{\epsilon}x, e^{-2\epsilon}y)$ is a symmetry flow for the differential equation $\frac{dy}{dx} = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$.



The green flow lines show the change in the blue solutions as ϵ changes. Two invariant solutions are shown in red.

- An *invariant solution* to a differential equation is one that is mapped to itself in the symmetry, i.e., it is invariant in the symmetry.
- In order to find invariant solutions to a symmetry, we use what are called canonical coordinates. Converting to canonical coordinates results in an equation that is much easier analyze and if we're lucky, solve.
- Once a solution is found for the transformed equation, we can easily transform back to the original coordinates using the definitions for our canonical coordinates.

2. Symmetry Generators

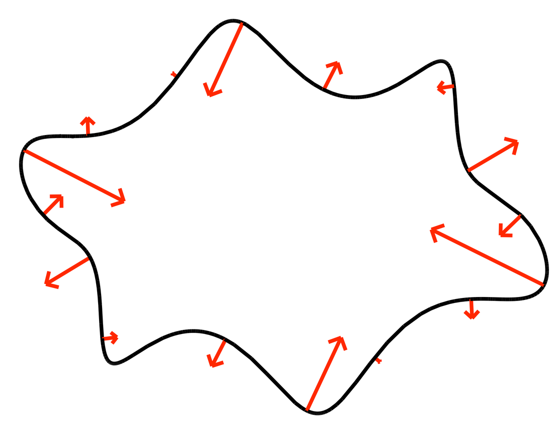
- Symmetries can be expressed in one of two ways
  - as $(\hat{x}, \hat{y})$ given as functions of the old coordinates (x, y) and a parameter ϵ
  - as a symmetry generator $X = \xi\partial_x + \eta\partial_y$ where ξ and η are functions of x and y defined by

$$\xi = \frac{d\hat{x}}{d\epsilon}\Big|_{\epsilon=0} \qquad \eta = \frac{d\hat{y}}{d\epsilon}\Big|_{\epsilon=0}$$

- All symmetries for a differential equation, $\frac{dy}{dx} = \omega(x, y)$ must satisfy what is known as the symmetry condition
  - The full symmetry condition is used with functions $\hat{x}$ and $\hat{y}$: $\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y})$.
  - For symmetry generators, we linearize this condition around $\epsilon = 0$.
- In order to find symmetries, we use the linearized condition because the linear equations that result are typically easier to solve.

3. Curve Shortening

- Curve shortening is a geometric evolution that when given a curve, the curve continually evolves based on the curvature [2].

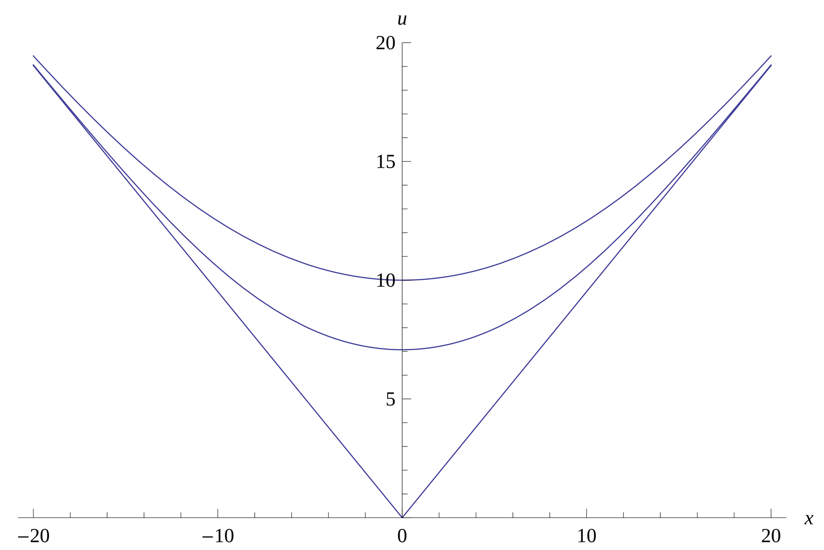


Velocity vectors for the curve shortening flow.

- This process is defined by assigning a velocity, equal to the curvature k, to each point on the curve $\vec{r}$ in the direction of the normal vector $\vec{N}$. Mathematically this is expressed as $\frac{\partial\vec{r}}{\partial t} = k\vec{N}$.
- For the curve shortening equation, the invariant solutions are the self similar solutions, the curves that maintain their form as they go through the process.
- We analyzed the curve shortening equation in two ways, first by looking directly at the case on the curve as the graph of a function (Section 4) and second by looking at the evolution of the curvature (Section 5).

4. Curve Shortening for the Graph of a Function

- As shown in [1], the first option of looking directly at the curve as the graph of a function $u(x)$ results in the differential equation $u_t = \frac{u_{xx}}{1+u_x^2}$.



An example of curve shortening for the graph of a function.

- Building off of Chou and Li's work, we looked for the invariant solutions for the symmetry generator $X = x\partial x + 2t\partial t + u\partial u$
- This resulted in the canonical coordinates $r = x/\sqrt{t}$ and $F(r) = u/\sqrt{t}$ in terms of which invariant solutions are determined by the differential equation

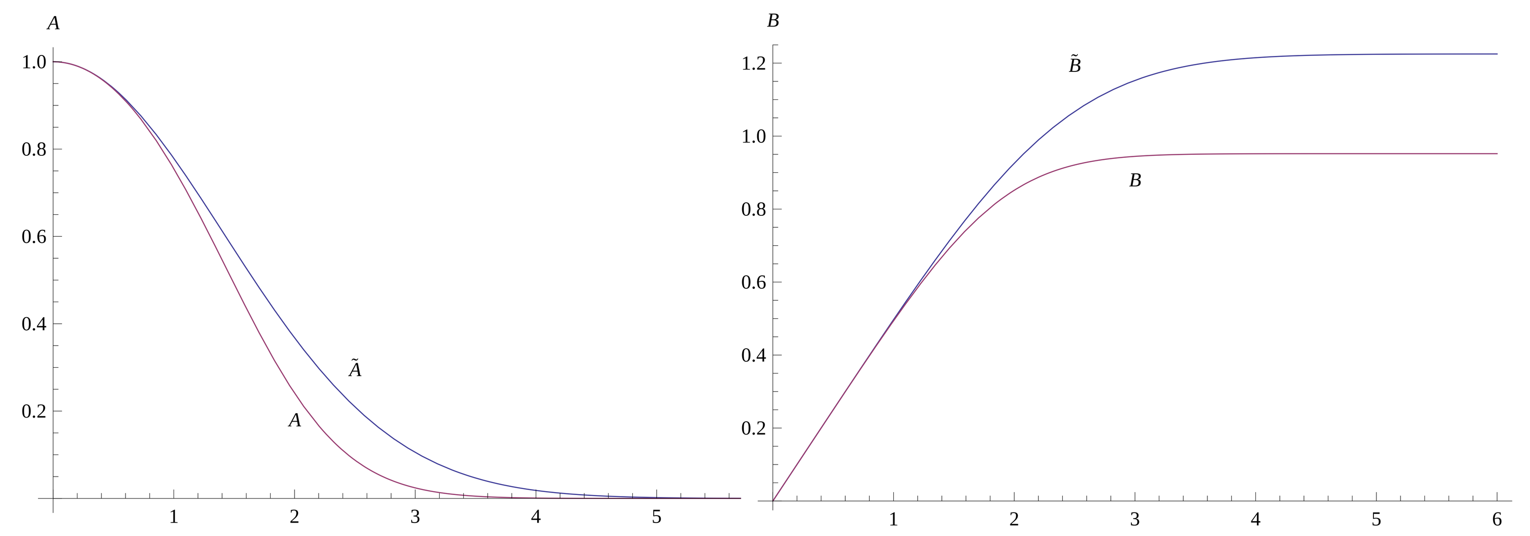
$$2F'' = (1 + F'^2)(F - rF').$$

- Solutions to this differential equation are not immediately apparent, so we broke it into a first-order system using the quantities $F' = B$ and $A = F - rB$ to get

$$A' = -\frac{r}{2}(F - rB)(1 + B^2) = -\frac{r}{2}A(1 + B^2)$$
$$B' = \frac{1}{2}(F - rB)(1 + B^2) = \frac{1}{2}A(1 + B^2)$$

- Again, solutions aren't immediately apparent, but we can find upper bounds on A and B to describe the evolution of F since $F = A + rB$
- Upper bounds on A and B are given by

$$A \le \tilde{A} = A_0 e^{\frac{-(1+B_0^2)}{4}r^2} \quad \text{and} \quad B \le \tilde{B} = \tan\left(\frac{A_0\sqrt{\pi}}{2\sqrt{1+B_0^2}}\,\mathrm{erf}\left(\frac{r}{2}\sqrt{1+B_0^2}\right) + \tan^{-1}(B_0)\right).$$



A (red) and the bound $\tilde{A}$ (blue) — *B (red) and the bound $\tilde{B}$ (blue)*

- Also of note is that at low values of A_0, the bounds become remarkably close to their respective functions
- Because the limit as r goes to infinity of $\tilde{A}$ is 0, the only term that has any effect on the limit of F is rB. Since B limits to a constant, F will be asymptotically linear.

5. Curve Shortening Applied to the Curvature

- For the curve shortening system, our independent variables are time t and the arbitrary parameter p. The dependent variables are the curvature k and $v = |\frac{\partial\vec{r}}{\partial p}|$ where $\vec{r}$ is the vector valued function for the curve.
- The symmetry generator takes the form $X = \xi\partial_p + \tau\partial_t + \chi\partial_k + \eta\partial_v$
- We are able to deduce the following system of differential equations from the original curve shortening equation.

$$\frac{\partial k}{\partial t} = \frac{1}{v^2}\frac{\partial^2 k}{\partial p^2} - \frac{1}{v^3}\frac{\partial v}{\partial p}\frac{\partial k}{\partial p} + k^3$$
$$\frac{\partial v}{\partial t} = -k^2 v$$

- From the linearized symmetry condition, we get a system of 31 determining equations. From this system, we are able to deduce

$$\xi = C(p), \quad \tau = -2c_1 t + c_2, \quad \chi = c_1 k, \quad \text{and} \quad \eta = -v(C'(p) + c_1)$$

  where c_1 and c_2 are constants and C is any differentiable function.
- The above generator describes all possible symmetries for our system, so the next step was to find invariant solutions for particular generators. The generator that we analyzed was $X = p\partial p + 2t\partial t - k\partial k$.
- This generator results in the canonical coordinates $r = \frac{p}{\sqrt{t}}$, $G(r) = v$, and $H(r) = k\sqrt{t}$
- Once completely converted to canonical coordinates, the system turns into the following:

$$G' = \frac{2H^2G}{r} \qquad H'' = \frac{-G^2}{2}(rH' + H - 2H^3) + \frac{2H^2H'}{r}$$

- Though made difficult with the factor of r^{-1}, the next step would be to analyze these equations. However, this was beyond the scope of this project for the summer.

References

[1] K.S. Chou and G.X. Li. Optimal systems and invariant solutions for the curve shortening problem. *Communications in Analysis and Geometry*, 10:241-274, 2002.

[2] M.E. Gage and R.S. Hamilton. The heat equation shrinking convex plane curves. *Journal of Differential Geometry*, 23:69-96. 1986.

[3] Peter E. Hydon. *Symmetry Methods for Differential Equations*, Cambridge University Press, New York, 2000.