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The ray form of Newton's law of motion

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Through the use of the optical-mechanical analogy, Newton's law of motion may be cast into the same form as the equation for the ray in the geometrical optics of gradient-index media. The resulting equation is called the ray form of Newton's law of motion. The same equation may be derived by taking the geometrical optics limit of quantum mechanics. The ray form of Newton's law of motion is derived in three different ways and is applied in the solution of several problems.

I. INTRODUCTION

One may express the optical-mechanical analogy in several different forms. In Hamilton's original formulation[1,2] of the analogy, both mechanics and geometrical optics were couched in terms of the theory of the characteristic function. Mechanics and geometrical optics met, as it were, in rather high and difficult terrain.

There do exist, however, more elementary formulations of the optical-mechanical analogy. For example, it is possible[3] to cast the equation governing the optical ray into the form $\mathbf{F}=m\mathbf{a}$. The advantage of such a formulation is that techniques and strategies familiar from mechanics are immediately applicable in the realm of geometrical optics.[4]

In this article, we pursue the opposite course. We derive an equation for the trajectory of a particle which takes on the form of the standard equation for the ray in geometrical optics. We discuss the meaning of the equation and show how it may be applied to solve a number of problems in mechanics.

II. THE RAY FORM OF NEWTON'S LAW OF MOTION

A. First derivation

In geometrical optics, a frequently used equation for the ray[5,6] is

$$\nabla n=\frac{d}{ds}\left(n\frac{d\mathbf{x}}{ds}\right),\quad \text{(geometrical optics)},\qquad (1)$$

where $\mathbf{x}$ is the position of a point on the ray, ds is the element of arc length along the ray, and $n(\mathbf{x})$ is the index of refraction, which is a function of position. This differential equation may be integrated—either exactly, if the function $n(\mathbf{x})$ is simple enough, or numerically—to obtain the shape of the ray.

We shall now write down the corresponding equation for the trajectory of a particle in classical mechanics. Equation (1) can be derived in geometrical optics by applying the calculus of variations to Fermat's principle

$$\delta\int(n/c)ds=0,\quad \text{(geometrical optics)}.\qquad (2)$$

δ denotes a variation of the integral produced by a variation of the path of integration between two fixed points in space. c is the speed of light in vacuum. Thus c/n is the speed of light. The analogous principle in mechanics is Maupertuis' principle

$$\delta\int mv\,ds=0,\quad \text{(particle mechanics)}.\qquad (3)$$

m is the mass of a particle and $v(\mathbf{x})$ is the particle's speed $(=|\mathbf{v}|)$, considered to be a function of its position alone. The rules governing the variation of the integral are the same as for Fermat's principle. By comparison of Eq. (2) and Eq. (3), it is clear that we may pass over from geometrical optics to point-particle mechanics by the transcription[7]

$$n/c\Rightarrow mv.$$

This transcription applies whenever the force on the particle is derivable from a potential that depends on the position alone. Thus we may immediately write down for mechanics[8]

$$\nabla v=\frac{d}{ds}\left(v\frac{d\mathbf{x}}{ds}\right),\quad \text{(particle mechanics)}.\qquad (4)$$

Equations (4) and (1) are of perfectly similar form. However, Eq. (4) can be simplified if we note that $d\mathbf{x}/ds$ is a unit vector tangent to the trajectory,[9] i.e., in the direction of $\mathbf{v}$. Thus we have

$$\nabla v=d\mathbf{v}/ds,\quad \text{(particle mechanics)}.\qquad (5)$$

This we shall call the *ray form of Newton's law of motion.*

B. Second derivation

The ray form of Newton's law of motion may also be derived easily from straightforward mechanical principles. We begin with Newton's law of motion in the form

$$-\nabla U=m\,d\mathbf{v}/dt,$$

where the potential energy U is the difference between the total energy and the kinetic energy

$$U=E-mv^2/2.$$

Upon substitution for U, we have

$$v\nabla v=\frac{d\mathbf{v}}{dt}=\frac{d\mathbf{v}}{ds}\frac{ds}{dt}.$$

And thus

$$\nabla v=d\mathbf{v}/ds,$$

as just obtained by use of the optical-mechanical analogy.

C. Third derivation

We begin again with the central equation [Eq. (1)] of geometrical optics, but express it in terms of the wave number k rather than the index of refraction ($n=ck/\omega$, where ω is the angular frequency). Thus

$$\nabla k = \frac{d}{ds}\left(k \frac{d\mathbf{x}}{ds}\right). \tag{6}$$

Equation (6) is most familiar in the context of geometrical optics, but it is in fact far more general. As mentioned already, Eq. (6) can be derived by applying the calculus of variations to Fermat's principle [Eq. (2)].[10] Thus, a *ray* is a path of integration that makes the value of the Fermat integral stationary against small variations in the shape of the path. Physically, the ray may be considered to result from constructive interference of the disturbances propagated along a bundle of closely neighboring virtual paths. Fermat's principle must therefore apply in "the geometrical optics limit" to *any* wave disturbance that obeys the principle of superposition. In particular, Eq. (6) must apply to the matter waves of quantum mechanics. That is, Eq. (6) is also the differential equation for the classical trajectory of a material particle: the particle trajectory is the "ray," in the short-wavelength limit of quantum mechanics. To apply Eq. (6), which is very general, to the particular case of mechanics, we use the de Broglie relation

$$m\mathbf{v} = \hbar\mathbf{k}, \tag{7}$$

where $\hbar$ is Planck's constant divided by 2π. When Eq. (7) is used to replace k by v in Eq. (6) we immediately obtain Eq. (5), the ray form of Newton's law of motion.[11]

D. Discussion

Equation (5) may at first glance seem almost an identity, a near tautology. But it is not: It is merely concise. The first derivation shows that Eq. (5) is a particularly simple expression of the optical-mechanical analogy: mechanics takes on the form of the central equation of geometrical optics. The second derivation shows that Eq. (5) is equivalent to Newton's law of motion when the force may be derived from a velocity-independent potential. The third derivation shows that Eq. (5) succinctly expresses classical mechanics as the short-wavelength limit of wave mechanics.

For a given particle, with fixed energy E, the speed v appearing on the left side of Eq. (5) is to be thought of as a function of position. Imagine an ensemble of identical particles, all with same energy E, moving in an externally produced field. The speed of any one of the particles, at any point in space, depends only on its position $\mathbf{x}$. That is, we characterize the field of influence in which the particle moves by the function $v(E,\mathbf{x})$, rather than by $U(\mathbf{x})$. The left side of the equation thus represents the influence of the external world, and the right side represents the response of the particle. In words, the equation states that the change in velocity per unit distance traveled is equal to the gradient of the speed function.

III. SOME APPLICATIONS

The meaning of Eq. (5) can be made clearest by demonstrating its use in the solution of concrete problems. Every formulation of mechanics will be found well suited to some applications and workable but clumsy for others. The problems below can, of course, all be solved by other means. These problems have been selected because they illuminate essential features of the ray approach to mechanics and because they can be solved with special ease using this approach.

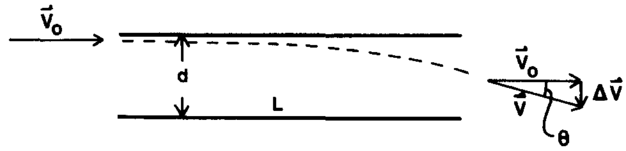

Fig. 1. A particle with initial velocity $\mathbf{v}_0$ is deflected through angle θ while traveling through a parallel-plate capacitor.

A. Uniform circular motion

Let a particle be moving on a circle of radius r at constant speed v. Consider an infinitesmal arc ds of the circle, which subtends a central angle $d\theta$. While the particle is moving along ds, its velocity vector $\mathbf{v}$ rotates through the same angle $d\theta$, thus producing a change in velocity $d\mathbf{v}$. In the usual way, we invoke the similarity of two triangles: a triangle in coordinate space with two long sides r and a short side ds, and a triangle in velocity space with two long sides v and a short side $|d\mathbf{v}|$. The similarity of the triangles leads immediately to

$$\left|\frac{d\mathbf{v}}{ds}\right| = \frac{v}{r}. \tag{8}$$

The direction of $d\mathbf{v}/ds$ is towards the center of the circle. This is the kinematic relation that expresses uniform circular motion. It contains no physics. (It is equivalent to $a = v^2/r$, where a is the acceleration.) To put in the physics, we use the ray form of Newton's law of motion, Eq. (5), and thus obtain

$$|\nabla v| = v/r, \tag{9}$$

which is the dynamical condition for uniform circular motion (equivalent to $F = mv^2/r$, where F is the centripetal force).[12] The direction of ∇v is radially inwards towards the center of the circle.

B. Example involving evaluation of the speed gradient

Problem: A particle is traveling at speed v in a circular orbit of radius r about the center of the Earth. At a point on the orbit, the particle encounters a (perfectly elastic) barrier set at a 45° angle to the direction of motion. The particle is reflected radially downward towards the center of the Earth. By how much does the particle's speed change in the first meter of fall?

Solution: The circular orbit is characterized by Eq. (9); thus, the magnitude of the speed gradient is known at radius r. Since the value of the gradient does not change sensibly in only 1 m, the particle's change in speed in the 1-m fall is

$$\Delta v = (v/r) \times 1 \text{ meter}. \tag{10}$$

C. Example involving approximate evaluations of both ∇v and $d\mathbf{v}/ds$

Problem: An electron with speed v_0 enters a region of uniform electric field as shown in Fig. 1. The field is provided by a parallel plate capacitor of width L and plate separation d, with $d \ll L$. The electron enters the field near the top plate and its initial line of motion is parallel to the

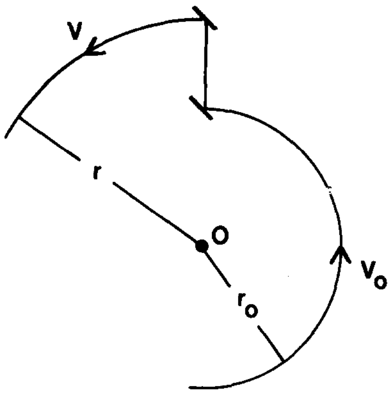

Fig. 2. A particle, initially in a circular orbit about a force center, is sent into a higher circular orbit by means of two reflections.

plates. It is desired to use the capacitor field to deflect the electron through a small angle θ. What is the minimum possible plate separation d?

Solution: The electron emerges from the capacitor with velocity $\mathbf{v}$, as shown in the figure. The change in velocity $\Delta\mathbf{v}$ is perpendicular to the original line of motion and its magnitude is $v_0 \tan\theta$. Because $d \ll L$, the distance traveled by the electron between the plates is essentially L. Thus the magnitude of the right-hand side of Eq. (5) is[13]

$$|d\mathbf{v}/ds| \simeq v_0 \tan\theta/L. \tag{11}$$

The left-hand side of Eq. (5) may be evaluated as follows. The gradient of the speed is entirely in the vertical direction. The speed changes by $(v-v_0)$ as the particle falls through a vertical distance d. Thus

$$\begin{aligned}|\nabla v| &\simeq (v-v_0)/d \\ &= [(v_0^2+v_0^2\tan^2\theta)^{1/2}-v_0]/d \\ &\simeq v_0\tan^2\theta/(2d).\end{aligned} \tag{12}$$

Setting the right sides of Eq. (11) and Eq. (12) equal to one another immediately gives

$$d \simeq (L/2)\tan\theta, \tag{13}$$

the minimum plate separation that will allow deflection through angle θ.

Equation (13) can, of course, be derived by other means. In the present context, it underscores an important feature of Eq. (5): The derivatives on the left-hand side are derivatives of the scalar v, while the derivative on the right-hand side is a derivative of the vector $\mathbf{v}$. This, in essence, is the source of the vital 1/2 in Eq. (13).

D. Example involving integration of the speed gradient

Problem: A particle is in uniform circular motion (speed v_0 and radius r_0) about a force center O located at the center of the circular orbit. The nature of the force law is unknown. However, whenever the particle is deflected by elastic reflection through two 90° turns (as in Fig. 2), it is always found that the new orbit also involves uniform circular motion about the same center, but at a new radius r. Find the speed v in the new orbit. Also determine r_{max}, the largest distance from O that can be attained in this manner.

Solution: Given the initial orbit parameters, r_0 and v_0, uniform circular motion is possible at every r. So, by Eq. (9),

$$\frac{dv}{dr} = -\frac{v}{r}. \tag{14}$$

The minus sign indicates explicitly that the direction of ∇v is inwards towards O. A simple integration gives

$$v = k/r,$$

where k is a constant of integration. Since this relation must also be satisfied by the initial orbit, $k = v_0 r_0$. So the speed v in the new orbit of radius r is

$$v = v_0 r_0/r. \tag{15}$$

The maximum distance r_{max} from O that is attainable by elastic reflection of the particle is clearly infinite. For we have $v \to 0$ as $r \to \infty$. Thus, given that the particle is in uniform circular motion at one radius, a circular orbit is possible at any radius without modification of the particle's energy. (In working this problem, it was not necessary to determine the form of the potential energy function. But because many readers will be happier with $U(r)$ than with $v(r)$, we will add that it turns out $U \propto -r^{-2}$. That is, the situation described in the problem implies an inverse-cube attractive force.[14]

This problem may, of course, be worked out from the beginning by other methods. The reader is encouraged to try an alternative solution, starting, say, from $mv^2/r = dU/dr$, then using conservation of energy. This solution is substantially longer than the one presented above.

IV. CONCLUSION

Equation (5), the ray form of Newton's law of motion, illustrates the optical-mechanical analogy at an elementary level and reveals Newtonian dynamics as the geometrical optics limit of wave mechanics. On the practical side, Eq. (5) can be of use in solving certain kinds of mechanics problems—usually problems in which a time description is not required. Equation (5) therefore constitutes one more tool that may be added to the mechanics tool kit. The utility of Eq. (5) is not terribly great, however, because of the inconvenience that often attaches to the use of the arc length as an independent variable. Indeed, the advantages of the familiar $F=ma$ approach over Eq. (5) point the way to an important lesson for geometrical optics: the "$F=ma$" formulation of geometrical optics[15] has significant practical advantages over the traditional textbook formulation based on Eq. (1). The ray form of Newton's law of motion is therefore likely to be of most use in instruction as a means of illustrating the connections between optics and classical mechanics, as well as between classical mechanics and wave mechanics.

[1] W. R. Hamilton, "On a general method of expressing the paths of light, and of the planets, by the coefficients of a characteristic function," Dublin University Review (1833), pp. 795–826; reprinted in: *The Mathematical Papers of Sir William Rowan Hamilton*, 3 Vols., edited by A. W. Conway *et al.* (Cambridge University, Cambridge, 1931–1967), Vol. 1, pp. 311–332.

[2] For an excellent summary of Hamilton's work in mechanics and in geometrical optics, see T. L. Hankins, *Sir William Rowan Hamilton* (Johns Hopkins, Baltimore, 1980), pp. 61–87, 181–198.

[3] J. Evans and M. Rosenquist, "'$F=ma$' optics," Am. J. Phys. **54**, 876–883 (1986).

[4]For another example, see T. Sekiguchi and K. B. Wolf, "The Hamiltonian formulation of optics," Am. J. Phys. **55**, 830–835 (1987). The "Hamiltonian formulation" of the title refers not to Hamilton's formulation of geometrical optics, but to the use of equations in geometrical optics that resemble Hamilton's equations in dynamics.

[5]M. Born and E. Wolf, *Principles of Optics*, 6th ed. (Pergamon, Oxford, 1980), pp. 101–132.

[6]M. V. Klein, *Optics* (Wiley, New York, 1970), pp. 29–31.

[7]This transcription constitutes only a part of the complete optical-mechanical analogy. It suffices for the present discussion, which is concerned only with the *shape* of the particle trajectory. If we also wish to consider the time development of the motion, we complete the analogy by means of the transcription *optical action* ⇒ *time*. Thus, the light progresses in action along its ray in the same way that the mechanical particle progresses in time along its trajectory. See J. Evans, "Simple forms for equations of rays in gradient-index lenses," Am. J. Phys. **58**, 773–778 (1990), p. 774.

[8]For the special case of particles of zero total energy, Eq. (4) can be written in the form $\nabla U^{1/2}=(d/ds)[U^{1/2}(d\mathbf{x}/ds)]$, where $U(\mathbf{x})$ is the potential energy. See J. A. Arnaud, "Analogy between optical rays and nonrelativistic particle trajectories: A comment," Am. J. Phys. **44**, 1067–1069 (1976). However, we shall stick to Eq. (4), which is more general.

[9]$d\mathbf{x}/ds=(d\mathbf{x}/dt)(dt/ds)=\mathbf{v}/v$. The optical equation [Eq. (1)] corresponding to Eq. (4) can be simplified in a similar way. The optical counterpart of Eq. (5) is $\nabla n=d(\mathbf{u}n)/ds$, where $\mathbf{u}$ is a unit vector tangent to the ray. See Ref. 6, p. 30.

[10]See, for example, Ref. 3, p. 877.

[11]For a derivation of the familiar form of Newton's law of motion directly from Fermat's principle, see M. Rosenquist and J. Evans, "The classical limit of quantum mechanics from Fermat's principle and the de Broglie relation," Am. J. Phys. **56**, 881–882 (1988).

[12]The two equations derived in this paragraph can also be obtained by simple manipulations of $a=v^2/r$ and $|\nabla U|=mv^2/r$. But the whole point, and most of the fun, is to derive them in the spirit of geometrical optics, i.e., thinking in terms of s rather than t.

[13]Neither ∇v nor $d\mathbf{v}/ds$ is really constant in this situation. However, because $|\Delta\mathbf{v}|\ll v_0$, both are approximately constant.

[14]This may be shown as follows. Because the particle may escape to infinity with zero speed, its energy is zero under the usual convention, so $U=-mv^2/2$. Then, upon substitution of Eq. (15), we have $U(r)=-cr^{-2}$, where the constant c may be expressed in terms of the initial orbit parameters as $c=mv_0^2r_0^2/2$.

[15]References 3 and 7.

Electrostatic field bounds for model dielectric configurations

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The normal component of an electric field near a convex shaped object of zero net charge and dielectric constant ϵ embedded in a vacuum with an asymptotically uniform electric field has an upper bound $(\epsilon/\epsilon_0)|\mathbf{E}_0|$, where ϵ_0 is the vacuum permittivity, and $\mathbf{E}_0$ is the asymptotic field. This limit may be closely approached in the vicinity of regions of high surface curvature.

I. INTRODUCTION

The shape and intensity of electric fields present near a dielectric object in an asymptotically uniform electric field $\mathbf{E}_0$ is a problem of considerable interest from both didactic and practical viewpoints. Many graduate level courses in classical electrodynamics begin with a discussion of electric fields and potentials near a variety of simple-shaped objects (spheres and cylinders). The mathematical problem involves solving Laplace's equation ($\nabla^2\Phi=0$) subject to various boundary conditions. If the dielectric shape falls into one of several simple categories and the coordinate system is suitably chosen, then Laplace's equation will be separable and the solution can be written in "closed form." This generally means that it conforms to one of a relatively small number of functions studied and cataloged in the 19th century or earlier which comprise a major part of the standard mathematical repertoire of graduate students of physics.

In all, there exist eleven separable coordinate systems for Laplace's equation in three dimensions.[1] Of these the most involved are ellipsoidal coordinates. The problem of a dielectric ellipsoid suspended in a vacuum within an asymptotically uniform electric field is a particularly interesting example since in addition to being the most complex case which is solvable in closed form,[2,3] it appears to illustrate the characteristic behavior of fields under much more general circumstances. The maximum field strength ($E_{\max}$) which can occur in this case is given by $(\epsilon/\epsilon_0)|\mathbf{E}_0|$ and is directed normal to the surface of the ellipsoid whose major axis is aligned with the asymptotic field $\mathbf{E}_0$. Here ϵ and ϵ_0 are the dielectric constants of the ellipsoid medium and vacuum, respectively. The limit $E_{\max}$ will be approached in the case of a highly elongated needle or "surfboard" shape, just outside the regions of maximum surface curvature.

It is of interest to ask whether it is possible to show that this simple expression really does constitute a limit for the maximum E field without restriction to one of the standard "solvable" shapes. The fact that this simple expression for $E_{\max}$ turns out to be a valid limit for convex dielectric shapes generally has considerable practical significance as a bound on field enhancement that may provide a point of electrical discharge formation in the initiation of arcs. By way of contrast, it is well-known that very high electric fields can be found near regions of high surface curvature in the case of conducting surfaces. The $E_{\max}$ limit may also

