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Optimal Control for a Class of Age Structured Dynamic Models

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Abstract

The mathematical theory of optimal control plays an important role in natural resource conservation efforts. While scalar models can be used effectively on many problems, age-structured models can provide a more nuanced modeling tool. This work examines a class of age-structured dynamic models with potential applications to age-targeted harvesting.

Elements of Optimal Control

Optimal control lies at the intersection of optimization and dynamical systems theory. Problems elements include:

- A state x(t) and a control u(t).
- A state equation

$$\frac{dx}{dt} = f(x,u,t), \qquad x(0) = x_0$$

- An objective function

$$\Gamma(u) = \int_0^T g(x,u,t)dt$$

- A constraint equation

$$u_{\min} \leq u(t) \leq u_{\max}$$

- Goal: Solve for x(t) and u(t), given state equation, objective, and constraints

Sample Scalar Problem

A classic example problem involves maximizing the time discounted revenue of a fishery under logistic growth and a simple harvest model. Specifically, if x(t) denotes fishery biomass and u(t) denotes fishing effort, the optimal control problem is to solve:

$$\min_{u(t)} \int_0^T e^{-\delta t}[p \cdot qx(t)u(t) - c \cdot u(t)]dt$$

subject to

$$\frac{dx}{dt} = rx(t)\left(1 - \frac{x(t)}{K}\right) - qx(t)u(t)$$

and

$$x(0) = x_0 \qquad 0 \leq u(t) \leq u_{\max}$$

Solution Techniques

1. Write down the Hamiltonian

$$H(x,u,t,\lambda) = f(x,u,t) + \lambda(t) \cdot g(x,u,t)$$

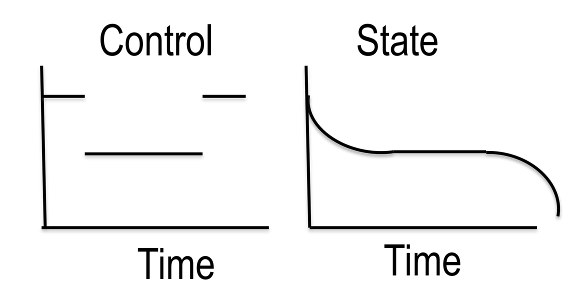
2. Derive optimality conditions

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x} \qquad \frac{\partial H}{\partial u} = 0 \qquad \lambda(T) = 0$$

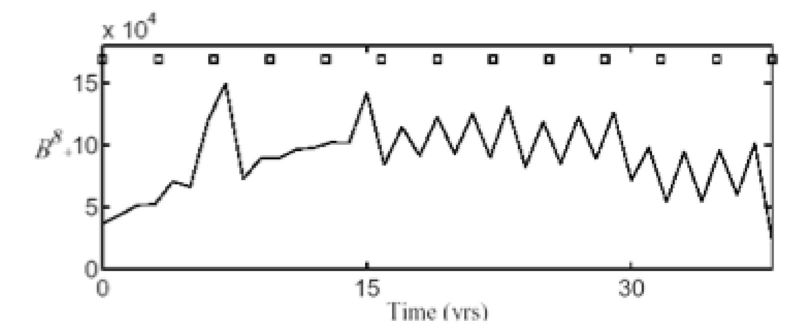
3. Use optimality conditions to solve for u(t) in terms of λ(t) and x(t)
4. Substitute into state equation and Lagrangian equation
5. Solve resulting system of differential equations

Solutions

Scalar model: "bang-bang" control



Discrete, age-structured, stochastic model: "Chattering" control



Motivation: Shearwaters

Some shearwater chicks are culled in order to be eaten.[1] Conservation efforts suggest a total ban on harvesting, but there may be more optimal approaches. We consider a two-class, age-structured ODE model with harvesting exclusively in the chick class. Objective is to maximize yield over time.

Model Formulation

The first model we considered mimics the form of the logistic growth model described above, with modifications made to better fit our case:

$$\max_E \int_0^T pqE(t)c(t) - E(t)\,dt$$
$$c'(t) = k_1 a(t)(1 - \frac{a(t)}{Q}) - k_2 c(t) - qE(t)c(t)$$
$$a'(t) = k_2 c(t) - k_3 a(t)$$

The next model is influenced by a Susceptible, Immune, and Recovered style problem, which is typically used for describing the spread and treatment of disease. In this scenario it expresses information about the variations in the adult and chick populations:

$$\max_E \int_0^T u^2(t) - c_1 c(t)\,dt$$
$$c'(t) = bN(t) - d_1 c(t) - u(t)c(t) - c_1 c(t)$$
$$a'(t) = u(t)c(t) - d_2 a(t)$$
$$N'(t) = (b - d_1 - d_2)N(t)$$

Conclusions and Future Work

We have created several continuous, age-structured dynamic Shearwater chick harvesting models. Numerical solutions are still in progress. It is an open question whether or not these solutions will exhibit "chattering" control.

Next steps include:

- Producing software that can solve these models numerically
- Subject numerical solutions to qualitative analysis
- Infer policy implications

References

[1] Dr. Peter Hodum, personal conversation
[2] Lenhart, S. and Workman, J. T., 2007, *Optimal Control Applied to Biological Models*, Taylor & Francis Group LLC, Boca Raton, Fl, 261 p.
[3] Russell, R. W., 1999, Comparative Demography and Life History Tactics of Seabirds: Implications for Conservations and Marine Modeling, *American Fisheries Society Symposium*, v. 23, p. 51-76.