11-1-1997

Answer To Question #55. Are There Pictorial Examples That Distinguish Covariant And Contravariant Vectors?

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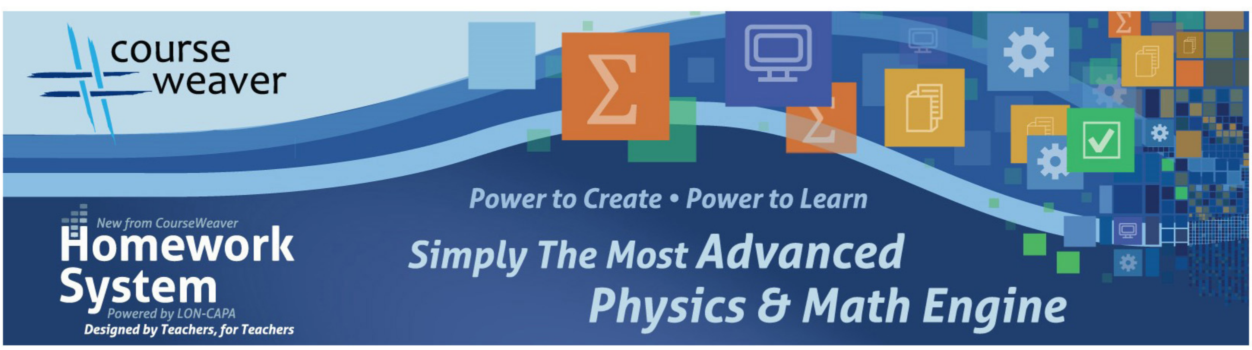
Citation: American Journal of Physics **65**, 1039 (1997); doi: 10.1119/1.18736
View online: http://dx.doi.org/10.1119/1.18736
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/65/11?ver=pdfcov
Published by the American Association of Physics Teachers



Answer to Question #55. Are there pictorial examples that distinguish covariant and contravariant vectors?

It is important to distinguish[1] between covariant and contravariant components of a vector whenever we deal with nondiagonal metric tensors, in fact, whenever the metric tensor is not the identity matrix. It is possible to construct informative pictorial examples even in the context of Euclidean plane geometry, if we choose to work with nonorthogonal axes.

Consider two axes, x and y, inclined at an angle α. An arbitrary vector $\mathbf{V}$ in the plane may be resolved into x and y components. But since the axes are not orthogonal, it is not so clear what we want to mean by "components of a vector." Two different definitions might be sensible. First, we might mean that $\mathbf{V}$ can be written in terms of line segments directed parallel to the unit vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{V}=V^x\mathbf{x}+V^y\mathbf{y}. \tag{1}$$

V^x and V^y are the contravariant components of $\mathbf{V}$. Second, we might mean that we should be able to pick off components by taking inner products of $\mathbf{V}$ with $\mathbf{x}$ and $\mathbf{y}$:

$$V_x=\mathbf{V}\cdot\mathbf{x}=V\cos\theta_x,\quad V_y=\mathbf{V}\cdot\mathbf{y}=V\cos\theta_y \tag{2}$$

(where θ_x and θ_y are the angles $\mathbf{V}$ makes with the x and y axes). V_x and V_y are the covariant components of $\mathbf{V}$. V^x, V^y and V_x, V_y are shown in diagram 1.

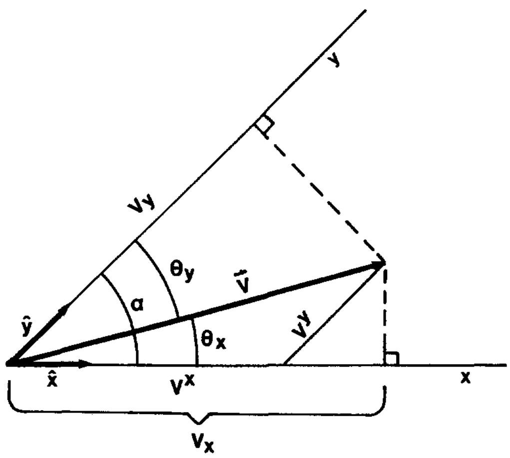


diagram 1

Let us construct the metric tensor. Taking the inner product of $\mathbf{x}$ with Eq. (1) we have

$$V_x=V^x+V^y\cos\alpha,$$

since $\mathbf{x}\cdot\mathbf{y}=\cos\alpha$. Similarly, taking the inner product of $\mathbf{y}$ with Eq. (1) gives

$$V_y=V^x\cos\alpha+V^y.$$

Thus the covariant components may be obtained from the contravariant components by the rule

$$V_i=g_{ij}V^j,$$

where i and j can each take on the values x and y, and where the repeated index is summed. The elements of the metric tensor are then

$$g_{ij}=\begin{pmatrix}1 & \cos\alpha\\ \cos\alpha & 1\end{pmatrix}.$$

It is easy to show that the inverse transformation is

$$V^i=g^{ij}V_j,$$

with

$$g^{ij}=\begin{pmatrix}\sin^{-2}\alpha & -\cos\alpha\sin^{-2}\alpha\\ -\cos\alpha\sin^{-2}\alpha & \sin^{-2}\alpha\end{pmatrix}.$$

The metric tensor has the usual properties. For example, $V^2=\mathbf{V}\cdot\mathbf{V}$ is an invariant, independent of the angle α between the axes. It may be calculated from

$$V^2=V^iV_i=V^ig_{ij}V^j=V_ig^{ij}V_j.$$

An easy way to see this is to begin from diagram 2,

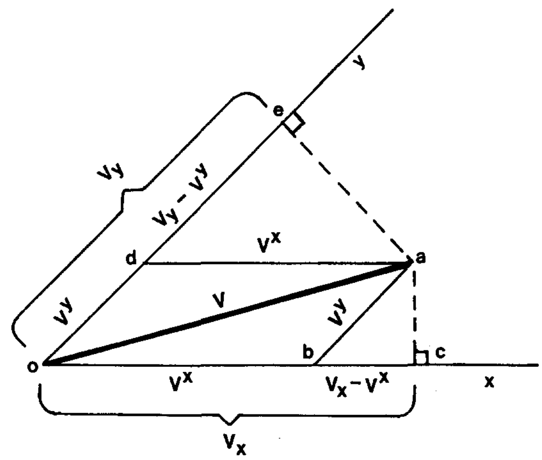


diagram 2

which is merely a more symmetric version of diagram 1. For the sake of having an expression of symmetric form, let us write

$$V^2=\tfrac{1}{2}[(oc)^2+(ca)^2+(oe)^2+(ea)^2].$$

Now,

$$oc=V_x$$

$$(ca)^2=(ab)^2-(bc)^2=(V^y)^2-(V_x-V^x)^2$$

$$oe=V_y$$

$$(ea)^2=(ad)^2-(de)^2=(V^x)^2-(V_y-V^y)^2.$$

With these substitutions, we find immediately

$$V^2=V^xV_x+V^yV_y.$$

Similarly, if $\mathbf{A}$ and $\mathbf{B}$ are two vectors, it may be shown that the invariant inner product (independent of α) is given by

$$\mathbf{A}\cdot\mathbf{B}=AB\cos\theta_{AB}=A^iB_i=A_iB^i,$$

where θ_{AB} is the angle between $\mathbf{A}$ and $\mathbf{B}$.

[1]D. Neuenschwander, Am. J. Phys. **65**(1), 11 (1997).

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